Black Hole Entropy in Loop Quantum Gravity

圈量子引力中的黑洞熵

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摘要

We give an account of the state of the art about black hole entropy in Loop Quantum Gravity. This chapter contains a historical summary and explains how black hole entropy is described by relying on the concept of isolated horizon, with an emphasis on different representations of its associated symmetry group. It continues with a review of the combinatorial methods necessary to understand the behavior of the entropy as a function of the area and concludes with a discussion of the nature of the quantum horizon degrees of freedom that account for the black hole entropy and the related issue of the fixing of the Immirzi parameter.

本文介绍圈量子引力中黑洞熵的研究现状。本章包含历史梳理，讲解了如何以孤立视界概念为基础描述黑洞熵，重点介绍了其关联对称群的不同表示方式。接着本文回顾了理解熵随面积变化规律所需的组合方法，最后讨论了贡献黑洞熵的量子视界自由度的本质，以及与 Immirzi 参数确定相关的问题。

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Introduction

引言

Black holes represent classical solutions of Einstein's equations of General Relativity, and they correspond to a final stage of isolated gravitational collapse. Starting on the late 60s, the investigation of black hole (BH) physics has been full of surprises, and it revealed intriguing properties which turned these objects into a perfect arena to test any candidate theory of Quantum Gravity. This chapter is devoted to the application of the Loop Quantum Gravity (LQG) formalism to a statistical mechanical treatment of BH microscopic degrees of freedom (DOF) in order to derive their semi-classical and continuum properties within a quantum gravity approach. The main focus is the derivation of the Bekenstein-Hawking entropy formula

黑洞是爱因斯坦广义相对论方程的经典解，对应孤立引力坍缩的最终阶段。自上世纪 60 年代末起，黑洞物理研究硕果累累，发现了诸多引人关注的性质，使得黑洞成为检验各类量子引力候选理论的绝佳场所。本章将运用圈量子引力 (LQG) 形式体系，对黑洞的微观自由度 (DOF) 进行统计力学处理，以在量子引力框架下推导出黑洞的半经典与连续性质。我们的核心目标是推导贝肯斯坦-霍金熵公式

$$S = \frac{k_B A}{4\ell_P^2}, \tag{1}$$

where (in units $c = 1$) k_B is the Boltzmann constant, $\ell_P = \sqrt{G_N \hbar}$ is the Planck length, and A is the BH horizon area. In fact, right after the analogy between the laws of BH physics and those of ordinary thermodynamics was found in [1], Bekenstein [2] argued that a notion of entropy could be associated with a BH, namely $S = \alpha A/\ell_P^2$, where α is a dimensionless, undetermined constant (from now on we set $k_B = 1$). Plugging this proposal into the BH first law

其中 (单位为 $c = 1$) k_B 是玻尔兹曼常数，$\ell_P = \sqrt{G_N \hbar}$ 是普朗克长度，A 是黑洞视界面积。事实上，文献 [1] 发现黑洞物理定律与普通热力学定律的类比后，贝肯斯坦随即在文献 [2] 中提出，熵的概念可以对应到黑洞，即熵为 $S = \alpha A/\ell_P^2$，其中 α 是无量纲的未定常数 (下文我们将其设为 $k_B = 1$)。将该假设代入 BH 第一定律

$$\delta M = \frac{\kappa_H}{8\pi G}\delta A + \Phi_H \delta Q + \Omega_H \delta J, \tag{2}$$

relating different nearby stationary BH spacetimes, immediately leads to a notion of temperature proportional to the horizon surface gravity κ_H , namely $T_H = \hbar\kappa_H/(8\pi\alpha)$ – in (2) Φ_H is the electrostatic potential at the horizon and Ω_H the angular velocity of the horizon.

该定律描述相邻稳态黑洞时空之间的关系，代入后可直接得到黑洞温度与视界表面引力 κ_H 成正比的结论，即 (2) 式中的 $T_H = \hbar\kappa_H/(8\pi\alpha)$ –，其中 Φ_H 是视界处的静电势，Ω_H 是视界的角速度。

At the classical level though, the temperature of a BH is absolute zero since nothing can escape once inside the horizon. By using methods from quantum field theory in curved spacetimes, Hawking arrived at his famous discovery [3] of BH radiance with a black body spectrum at temperature

但在经典层面，黑洞的温度为绝对零度，因为任何物体进入视界后都无法逃逸。霍金利用弯曲时空量子场论方法，得到了著名的发现 [3]: 黑洞会向外辐射，辐射谱是温度满足下式的黑体谱

$$T_H = \frac{\hbar\kappa_H}{2\pi}, \tag{3}$$

thus confirming the physical nature of surface gravity as the temperature of a BH and fixing $\alpha = 1/4$, which yields the entropy formula (1).

这一结果证实了表面引力就是黑洞温度的物理本质，并确定了 $\alpha = 1/4$ 的取值，由此得到了熵公式 (1)。

Although Hawking's derivation relies on a semi-classical regime, analyzing the scattering properties of a quantum test field on the background geometry of a large BH before and after the collapse dynamical phase, the final elegant expression (1) for the BH entropy involves the Planck length, i.e., the scale at which the quantum aspects of the gravitational field cannot be neglected any more. Therefore, a proper statistical mechanical understanding of the Bekenstein-Hawking formula can only be achieved within a quantum gravity description of the horizon microscopic DOF. It is recovering the exact numerical factor in front of the entropy leading term which would then require to go to the semi-classical regime of the given quantum gravity approach. Our goal is to show how these two steps can be completed within the LQG formalism (for other detailed reviews see [4-6]).

尽管霍金的推导建立在半经典近似之上——即分析大质量黑洞坍缩前后，量子测试场在背景几何上的散射性质，但黑洞熵最终的简洁表达式 (1) 中包含了普朗克长度，也就是引力场量子效应不可再被忽略的特征尺度。因此，只有通过量子引力对视界微观自由度进行描述，才能正确从统计力学角度理解贝肯斯坦-霍金公式。要得到熵领头项前的精确数值系数，需要进入给定量子引力方法的半经典区域。我们的目标是展示如何在 LQG 框架中完成这两步工作 (其他详细综述参见 [4-6])。

History of Black Hole Entropy in LQG

圈量子引力中黑洞熵的研究历史

The beginning of the investigation of BH entropy in LQG can be dated back to the pioneering work of Smolin [7], where an interplay between a topological QFT on the boundary and a non-perturbative quantum gravity description of the gravitational field in the bulk was proposed to describe the state space of a horizon and to provide evidence for the holographic hypothesis. Subsequently, Rovelli [8] defined a statistical mechanical system modeling a quantum non-rotating and non-charged horizon characterized by a macroscopic parameter given by the LQG eigenvalue of the horizon area. By identifying the BH entropy with the number of microstates of the horizon quantum geometry compatible with the macroscopic configuration, he obtained a result proportional to the area. Combining these results, Krasnov [9] proposed to use $SU(2)$ Chern-Simons (CS) theory to give a quantum mechanical description for the microscopic states of a large Schwarzschild BH. The interplay between LQG techniques and the CS formalism of the boundary theory led to a linear dependence of the entropy on the area, with a proportionality coefficient depending on the Immirzi parameter γ .

圈量子引力中黑洞熵研究的开端可追溯至斯莫林的开创性工作 [7]，该工作提出利用边界拓扑量子场论与 bulk 引力场的非微扰量子引力描述之间的相互作用，描述视界的态空间并为全息假说提供证据。随后，罗韦利 [8] 定义了一个统计力学系统，对量子非转动不带电视界建模，该视界由圈量子引力中视界面积的本征值给出的宏观参数刻画。通过将黑洞熵与和宏观构型相容的视界量子几何微态数对应，罗韦利得到了与面积成正比的结果。结合这些成果，克拉斯诺夫 [9] 提出使用 $SU(2)$ 陈-西蒙斯 (CS) 理论对大质量史瓦西黑洞的微观态给出量子力学描述。圈量子引力技术与边界理论的陈-西蒙斯形式体系结合，得到了熵对面积的线性依赖关系，其比例系数取决于伊米尔齐参数 γ 。

This framework for BH entropy computations in LQG was put on firm ground by the introduction of a quasi-local notion of horizon in equilibrium, with boundary conditions on the gravitational field specified only at a given inner boundary of spacetime. This is the notion of Isolated Horizon (IH) introduced in [10,11] (see also [12,13]), which provides a characterization of a static horizon but eliminates the need to have a knowledge of the complete spacetime (like in the case of event horizons); this allows for different bulk dynamical configurations which might be more physically relevant (or even necessary) in certain quantum gravity scenarios. At the same time, the IH framework is restrictive enough to recover the zeroth and first laws of BH thermodynamics [14, 15].

这种圈量子引力中计算黑洞熵的框架，因引入了平衡态视界的准局部概念而拥有了坚实基础，该框架仅需在时空给定内边界上指定引力场的边界条件。这就是文献 [10,11] 中引入的孤立视界 (IH) 概念 (另见 [12,13])，它刻画了静态视界，同时不需要知晓完整时空的信息 (这是事件视界要求的条件)；这允许存在不同的 bulk 动力学构型，在某些量子引力场景中这些构型可能更具物理相关性 (甚至是必需的)。同时，孤立视界框架的约束性足以推导出黑洞热力学第零定律和第一定律 [14, 15]。

In the original analysis of the IH classical phase space [16], it was shown that the symplectic structure of general relativity in the first-order formalism acquires, through gauge fixing of the internal symmetry, a boundary term parametrized by a $U(1)$ CS connection. The quantization of this enlarged phase space was carried out in [17], by, respectively, applying LQG and CS techniques to the bulk and boundary Hilbert spaces. By coupling the two through the quantum imposition of IH boundary conditions for a large, fixed value of the horizon area, the linear behavior in the area of the leading term in the entropy was confirmed, as well as the need to fix γ to a specific numerical value so to recover the Bekenstein-Hawking entropy formula.

在对孤立视界经典相空间的最初分析 [16] 中，研究表明一阶形式下广义相对论的辛结构，通过内部对称性的规范固定，会得到一个由 $U(1)$ 陈-西蒙斯联络参数化的边界项。文献 [17] 完成了这个扩展相空间的量子化，分别对 bulk 希尔伯特空间和边界希尔伯特空间应用圈量子引力技术和陈-西蒙斯技术。对于视界面积取大的固定值的情况，通过量子化施加孤立视界边界条件将二者耦合，研究证实了熵主导项对面积的线性依赖关系，同时也证实了需要将 γ 固定为特定数值才能得到贝肯斯坦-霍金熵公式。

While the fixation of the Immirzi parameter through the BH entropy calculation was presented as a way to remove an ambiguity of the quantum formalism, such approach may seem unnatural. In fact, at the classical level the Immirzi parameter does not encode any physical ambiguity: Different sectors of γ simply amount to different canonical transformations in the phase space that do not affect physical observables. It is only at the quantum level that γ becomes a true ambiguity through its appearance in the spectrum of geometrical operators. Therefore, relying on a semi-classical calculation like the one explaining Hawking radiation, where gravity is treated classically, in order to select a specific value of the Immirzi parameter may be an indication that some ingredients are missing - see however section "Entropy DOF and the Immirzi Parameter" for alternative views on this topic. Nevertheless, the seminal papers [16, 17] started a rich investigation of the counting problem that led to the discovery of sub-leading logarithmic corrections (independent of γ) and revealed a discrete structure of the entropy functional for small values of the IH area [18-24]. It should be noticed that, although the majority of the papers on BH entropy in LQG use the standard area operator, it is actually possible to make use of other natural choices such as the flux-area operator [25]. Although qualitatively this works much in the same way as the usual approach, there are some interesting differences. The most obvious one being that the area spectrum consists of equally spaced eigenvalues.

虽然通过黑洞熵计算确定伊米尔齐参数被提出作为消除量子形式体系歧义的一种方法，但这种方法可能看起来并不自然。实际上，在经典层面伊米尔齐参数不蕴含任何物理歧义：γ 的不同区间仅对应相空间中不同的正则变换，这些变换不会影响物理可观测量。只有在量子层面，γ 才会因为出现在几何算符的谱中成为真正的歧义。因此，依赖半经典计算 (比如解释霍金辐射的这类将引力经典处理的计算) 来选定伊米尔齐参数的特定值，可能说明我们还缺少某些核心要素——关于这一议题的不同观点可见“熵自由度与伊米尔齐参数”章节。尽管如此，这些开创性论文 [16, 17] 开启了对计数问题的丰富研究，带来了次领头对数修正 (与 γ 无关) 的发现，并揭示出当孤立视界面积取小值时熵泛函存在离散结构 [18-24]。需要注意的是，虽然圈量子引力中绝大多数黑洞熵研究都使用标准面积算符，但实际上也可以采用其他自然的选择，比如通量面积算符 [25]。尽管定性来看这种方法和常规方法非常相似，二者仍存在一些有趣的差异，最明显的一点是面积谱由等间距本征值构成。

At the same time though, this detailed investigation also led to the emergence of a second conflicting aspect. A logarithmic correction had been found before in [26], without relying on the IH formalism but by counting the conformal blocks of the $SU(2)$ Wess-Zumino-Witten model on a 2-sphere with punctures. A subleading term in the entropy was derived shortly after also by Carlip [27], relying on quite general symmetry considerations about the algebra of constraints in general relativity in the presence of a (local) Killing horizon; the appearance of a natural Virasoro subalgebra motivates the use of 2D CFT methods, previously proposed in [28], which, not surprisingly since in both cases CFT methods play a fundamental role, led to a logarithmic correction with the same numerical coefficient $-3/2$ as found in [26]. Carlip argued that, at least for non-rotating BHs, this coefficient might have a universal nature. However, this expectation was at clash with the result obtained with the $U(1)$ IH framework of [16, 17] , which yields a numerical coefficient of $-1/2$.

但与此同时，这项细致研究也引出了第二个存在矛盾的方面。此前文献 [26] 中就已发现对数修正，该修正不依赖孤立视界 (IH) 形式体系，而是通过对带 puncture 的二维球面上的 $SU(2)$ 韦茨-祖米诺-维滕模型的共形块计数得到。此后不久，卡利普也通过相当一般性的对称性分析推导出了熵的次领头项 [27]，他的分析基于存在 (局部) 基灵视界时广义相对论中约束代数的一般性质；天然出现的 Virasoro 子代数支持了此前文献 [28] 提出的二维共形场论 (CFT) 方法的适用性——由于两种情况中 CFT 方法都起到核心作用，因此不出意外，该方法得到的对数修正的数值系数和文献 [26] 中得到的 $-3/2$ 一致。卡利普提出，至少对非旋转黑洞而言，该系数可能具有普适性。然而，这一预期和 [16, 17] 的 $U(1)$ IH 框架得到的结果相冲突，后者给出的数值系数为 $-1/2$ 。

A way to understand such discrepancy is to observe that the algebra of the $U(1)$ IH quantum boundary conditions, relating the fluctuations of the boundary connection to those of the bulk fluxes defined on surfaces intersecting the horizon, does not preserve the Lie algebraic structure of the classical theory. It follows that, at the quantum level, one can impose only a subset of boundary conditions, leading to a slight overcounting of microstates. This problem with the gauge symmetry reduced model, together with an attempt to provide a more uniform treatment of the bulk and boundary DOF, in order to make contact with the original ideas of [7-9] and better understand the role of the Immirzi parameter, led to the development of an $SU(2)$ - invariant formulation of IHs [29-32]. These works clarified both the classical and the quantum frameworks, which, in the non-rotating case, allowed to show how the correct imposition of the quantum boundary conditions leads indeed to the -3/2 factor of the logarithmic correction [33].

理解这种差异的一种思路是: $U(1)$ IH 量子边界条件将边界联络的涨落与定义在相交于视界的截面上的体通量涨落联系起来，该条件对应的代数不保留经典理论的李代数结构。由此可知，在量子层面我们只能施加一部分边界条件，这导致对微态的计数略微偏多。规范对称约化模型存在的这一问题，加上人们试图对体和边界自由度给出更一致的处理，从而对接文献 [7-9] 的原始思想、更好地理解伊米尔齐参数的作用，最终推动了 IH 的 $SU(2)$ 不变表述的发展 [29-32]。这些工作厘清了经典和量子框架，在非旋转情形下得以证明: 正确施加量子边界条件后确实能得到对数修正的-3/2 因子 [33]。

Before reviewing the main technical aspects of the LQG BH entropy calculation, let us point out two important aspects that are common to these different technical approaches. The first one is the implementation of a ”weak holographic principle” (see, e.g., [34, 35] for an explicit formulation). This dates back to the first works on the subject [7, 8] (an interesting discussion about it can be found in [36]). Weak holography applied to the BH entropy counting implies that the relevant DOF are only those measurable by observers just outside the horizon. In the LQG literature, this principle has been applied by constructing the horizon density matrix by tracing over all the bulk DOF, both interior and exterior, and assuming the reduced density matrix to be maximally mixed. In this way, only the quantum horizon boundary DOF are considered. These are encoded in the structure of a single intertwiner (either $U(1)$ or $SU(2)$) between all the punctures created by the bulk links piercing the horizon. This intertwiner is assumed to be flat, namely the coarse graining over the bulk DOF is assumed to wash away all the information about the interior bulk curvature (see, however, [37] for an analysis of the holographic regime of LQG in the presence of bulk entropy) and the total flux across the boundary (in the Ashtekar-Lewandowski vacuum representation [38]) vanishes. There are valid arguments [6, 39, 40] why this notion of holography is the only one that can survive in a background-independent quantum gravity context, while stronger forms of holography may hold in a fixed background, semi-classical limit.

在回顾圈量子引力 (LQG) 黑洞熵计算的主要技术内容之前，我们先指出这些不同技术进路共通的两个重要方面。第一点是“弱全息原理”的实现 (具体表述可参见例如文献 [34, 35])。这一思路可以追溯到该领域的最早研究 [7, 8] (相关的有趣讨论可见文献 [36])。应用于黑洞熵计数的弱全息原理意味着，只有视界外侧附近观察者可测量的自由度才是相关自由度。在 LQG 文献中，这一原理的应用方式是: 对所有体自由度 (包括视界内外) 取迹构造视界密度矩阵，并假设约化密度矩阵是最大混态。通过这种方式，仅保留量子视界边界自由度。这些自由度被编码在单个交缠子的结构中 (既可以是 $U(1)$ ，也可以是 $SU(2)$)，交缠子位于刺穿视界的体链产生的所有 puncture 之间。该交缠子被假设为平坦的，即对体自由度的粗粒化会冲刷掉关于内部体曲率的所有信息 (不过对存在体熵时 LQG 全息区域的分析可参见文献 [37])，且穿过边界的总通量 (在阿西特卡-莱万多夫斯基真空表象 [38] 中) 为零。存在合理的论证 [6, 39, 40] 说明，为何这种全息概念是唯一能在背景无关量子引力框架下成立的全息概念，而更强形式的全息性可能仅在固定背景半经典极限下成立。

The second important aspect is that the LQG intertwiner construction of a quantum IH is purely kinematical. In fact, within the IH framework, it has been shown in [16] that, in order for the Hamiltonian time evolution (in the covariant phase space formalism) to be well defined, the lapse function smearing the Hamiltonian constraint of the canonical theory needs to vanish at the horizon. In other words, the bulk dynamics is switched off at the horizon (in this way the horizon area becomes a physical observable) and one assumes that for each IH boundary state there exists at least one physical bulk state compatible with it, which annihilates the Hamiltonian constraint. In this sense the quantum IH construction is purely kinematical.

第二个重要方面是，圈量子引力中对量子孤立视界的 intertwinner 构造纯粹是运动学的。实际上，在孤立视界框架内，文献 [16] 已证明: 为了让哈密顿时间演化 (在协变相空间形式中) 良定义，弥散正则理论哈密顿约束的移时函数需要在视界处为零。换言之，体动力学在视界处被关闭 (通过这种方式，视界面积成为一个物理可观测量)，且假设对每个孤立视界边界态，至少存在一个与之相容的物理体态，该体态湮灭哈密顿约束。从这个意义上说，量子孤立视界的构造纯粹是运动学的。

As a final recent addition to the toolbox used to study BHs in LQG, it is worth mentioning the recent work [41], where, instead of using a CS theory to describe the quantum DOF sitting at the BH horizon, the authors introduce an $SO(1,1)$ BF theory. The main consequence of this is the enlargement of the covariant phase space of the system, which can include now spacetime solutions with any isolated horizon as inner boundary.

作为圈量子引力黑洞研究工具中最新的补充，值得一提的是近期工作 [41]，作者在该工作中没有采用陈-西蒙斯理论描述黑洞视界处的量子自由度，而是引入了 $SO(1,1)$ BF 理论。这一改动的主要结果是扩大了系统的协变相空间，现在相空间可以包含以任意孤立视界为内边界的时空解。

Isolated Horizons

孤立视界

Isolated horizons replace the teleological notion of event horizon with a weaker and local definition involving the behavior of fields only at the horizon, while allowing us to derive laws analog to those of BH thermodynamics. Let us review the more relevant features of such a definition as provided in [11, 14, 15, 42] (for more extensive and technical reviews, see, e.g., [5,43]).

孤立视界用更弱的局域定义替代了目的论式的事件视界概念，该定义仅依赖视界处场的行为，同时还能让我们推导出类比黑洞热力学的定律。我们将梳理 [11, 14, 15, 42] 中给出的该定义的核心特征 (更全面的技术性综述参见例如文献 [5,43])。

Boundary Conditions

边界条件

Let us consider an asymptotically flat 4-manifold $\mathcal{M}$ with a metric g_{ab} of signature $(-,+,+,+)$ and a null hypersurface Δ of $(\mathcal{M}, g_{ab})$ with topology $\Delta = S^2 \times \mathbb{R}$. We denote by ℓ a future-directed null normal to Δ, ∇_a the derivative operator compatible with g_{ab} , and q_{ab} the degenerate intrinsic metric corresponding to the pull-back of g_{ab} to Δ . The IH boundary conditions require, first, that all the field equations and the stronger dominant energy condition hold at Δ . Furthermore, Δ is equipped with an equivalence class $[\ell]$ of null normals, whose members are related by a positive constant rescaling, and the expansion $\theta_{(\ell)}$ of any given null normal $\ell \in [\ell]$ has to vanish within Δ . These conditions are enough to guarantee that the horizon area of a given 2-sphere cross-section A_s is constant in time (no flux of matter nor gravitational radiation across Δ). Moreover, as implied by the Raychaudhuri equation, the null normal ℓ^a is also shear-free, so that the spacetime connection ∇_a induces a unique intrinsic connection D_a compatible with the induced metric q_{ab} .

The above conditions imply that the intrinsic metric q_{ab} is Lie dragged by ℓ^a , that is, $\mathcal{L}_\ell q_{ab} = 0$. The final restriction demands that the full intrinsic connection D_a be conserved along Δ , namely $[\mathcal{L}, D]|_\Delta = 0$. This condition allows one to define a notion of surface gravity κ_ℓ that is also constant along Δ for each $\ell \in [\ell]$ without the need to have a Killing field even in the proximity of Δ . We thus recover a generalized zeroth law of BH mechanics.

我们考虑一个渐近平坦的 4-流形 $\mathcal{M}$ ，它配备度规 g_{ab} ，号差为 $(-,+,+,+)$ ，流形 $(\mathcal{M}, g_{ab})$ 上存在一个零超曲面 Δ ，拓扑结构为 $\Delta = S^2 \times \mathbb{R}$ 。我们用 ℓ 表示指向未来、满足 Δ, ∇_a 的零法矢，用 g_{ab} 表示与 g_{ab} 相容的导数算符，q_{ab} 对应 g_{ab} 拉回至 Δ 得到的退化内蕴度规。孤立视界 (IH) 边界条件首先要求: 所有场方程和更强的主导能量条件在 Δ 上成立。此外，Δ 上配备一个零法矢的等价类 $[\ell]$ ，类内成员差一个正常数标度变换，且任意给定零法矢 $\ell \in [\ell]$ 的膨胀 $\theta_{(\ell)}$ 在 Δ 内必须为零。这些条件足以保证任意给定 2-球面截面 A_s 的视界面积不随时间变化 (没有物质或引力辐射穿过 Δ)。此外，由瑞乔杜里方程可知，零法矢 ℓ^a 也是无剪切的，因此时空联络 ∇_a 会诱导出与诱导度规 q_{ab} 相容的唯一内蕴联络 D_a 。上述条件意味着内蕴度规 q_{ab} 沿 ℓ^a 是李拖动的，即满足 $\mathcal{L}_\ell q_{ab} = 0$ 。最后一个约束要求完整内蕴联络 D_a 沿 Δ 守恒，即满足 $[\mathcal{L}, D]|_\Delta = 0$ 。该条件允许我们定义即使在 Δ 附近没有基灵场，对每个 $\ell \in [\ell]$ 而言，表面引力 κ_ℓ 也沿 Δ 保持常数。由此我们得到了推广的黑洞力学第零定律。

The isolated horizon boundary conditions provide a generalization of the Killing horizon concept that encompasses all the globally stationary BHs. The freedom left in the positive constant rescaling of the null normal ℓ^a reflects on the notion of surface gravity, whose normalization is thus undetermined. This leads to a family of first laws of IHs [15]. In the case of global Killing fields for asymptotically flat spacetimes, the same ambiguity is resolved by specifying the normalization of the Killing field at infinity. If required, one can select a unique first law for IHs, e.g., in the non-rotating case, by matching the surface gravity to that of stationary BHs.

孤立视界边界条件推广了基灵视界概念，涵盖了所有整体稳态黑洞。零法矢 ℓ^a 的正常数标度变换自由度会反映在表面引力的定义中，因此表面引力的归一化是不确定的，这就得到了一族孤立视界的第一定律 [15]。对于渐近平坦时空的整体基灵场，这种不确定性可以通过指定无穷远处基灵场的归一化来解决。如果需要，我们也可以为孤立视界选出唯一的第一定律，例如在非旋转情况下，让孤立视界的表面引力与稳态黑洞的表面引力匹配即可。

Phase Space

相空间

In order to construct the Hilbert space associated with a quantum IH and thus identify the DOF that account for its entropy, it is crucial to first study the IH phase space structure at the classical level. In order to do so, it proves convenient to employ covariant phase space methods. As the LQG quantization framework relies on the holonomy-flux algebra, we consider general relativity in its first-order formulation and, more precisely, we focus on the Einstein-Cartan-Holst gravity. In this case, the fundamental variables are represented by a tetrad coframe field composed of $\mathbb{R}^4$ -valued 1-forms e^I , with $I = 0, i$ and $i = 1, 2, 3$ labeling internal Lorentz indices, and a Lorentz connection $\omega^{IJ} = -\omega^{JI}$ with curvature $F^{IJ} = \mathrm{d}\omega^{IJ} + \omega^I{}_K \wedge \omega^{KJ}$.

为了构造与量子孤立视界 (IH) 关联的希尔伯特空间，并由此确定贡献其熵的自由度，首先在经典层面研究 IH 的相空间结构至关重要。为此，采用协变相空间方法十分方便。由于圈量子引力 (LQG) 量子化框架依赖全纯-流代数，我们考虑一阶表述的广义相对论，更准确地说，我们聚焦于爱因斯坦-嘉当-霍尔斯特引力。此时，基本变量由: $\mathbb{R}^4$ 值 1 形式场 e^I 构成的标架余标架场表示，其中 $I = 0, i$ 和 $i = 1, 2, 3$ 标记内部洛伦兹指标，另外还有洛伦兹联络 $\omega^{IJ} = -\omega^{JI}$ ，其曲率为 $F^{IJ} = \mathrm{d}\omega^{IJ} + \omega^I{}_K \wedge \omega^{KJ}$ 。

The Einstein-Cartan-Holst (ECH) Lagrangian is given by

爱因斯坦-嘉当-霍尔斯特 (ECH) 拉格朗日量由下式给出

$$L_{\mathrm{ECH}} = \frac{1}{2} E_{IJ} \wedge F^{IJ},\ E_{IJ}[e] := \left(* + \frac{1}{\gamma}\right)(e_I \wedge e_J), \tag{4}$$

where the duality operation is defined as $(*M)_{IJ} = \frac{1}{2}\varepsilon_{IJ}{}^{KL} M_{KL}$. The time gauge in an ADM-like decomposition of spacetime adopted in canonical LQG is imposed by demanding the component e^0 to be a time-like vector field normal to the Cauchy surface $\sum$ intersecting an IH Δ at a given 2-sphere cross-section $H = \Delta \cap \sum$. Upon this gauge fixing, the symplectic potential of the ECH formulation reads

其中对偶运算定义为 $(*M)_{IJ} = \frac{1}{2}\varepsilon_{IJ}{}^{KL} M_{KL}$ 。标准 LQG 中采用类 ADM 时空分解的时间规范，要求分量 e^0 是法向于柯西面 $\sum$ 的类时矢量场，该柯西面在给定二维球面截面 $H = \Delta \cap \sum$ 处与孤立视界 Δ 相交。完成该规范固定后，ECH 表述的辛势可写为

$$\kappa \Theta_{\mathrm{ECH}} = \frac{1}{\gamma} \int_{\sum} E_i \wedge \delta A^i, \tag{5}$$

where $\kappa = 16\pi G, E^i := \varepsilon^i{}_{jk} e^j \wedge e^k$, and $A^i := \Gamma^i + \gamma K^i$ is the SU(2) real Ashtekar connection. Here $K^i := \omega^{0i}$ is the extrinsic curvature of $\sum$, while $\Gamma^i = -\frac{1}{2}\varepsilon^{ijk}\omega_{jk}$ is the spin connection such that Cartan's equation $\mathrm{d}_\Gamma e^i = 0$ holds. The symplectic potential (5) tells us that the gravitational flux E^i and the real Ashtekar connection A^i are canonical pairs in the bulk $\sum$ if we start with the ECH formulation of gravity. However, sticking to a connection variable in the bulk, we see that the ECH Lagrangian does not give rise to any corner term on the IH cross-section H . Such a term would appear if we revert to vector-like variables (E^i, K^i) in the bulk; in fact, using the properties of Γ^i , it can be shown that (5) can be recast in the form

其中 $\kappa = 16\pi G, E^i := \varepsilon^i{}_{jk} e^j \wedge e^k$ ， 且 $A^i := \Gamma^i + \gamma K^i$ 是实 Ashtekar 联络 SU(2) 。此处 $K^i := \omega^{0i}$ 是 $\sum$ 的外曲率，$\Gamma^i = -\frac{1}{2}\varepsilon^{ijk}\omega_{jk}$ 是满足嘉当方程 $\mathrm{d}_\Gamma e^i = 0$ 的自旋联络。若我们从引力的 ECH 表述出发，辛势 (5) 表明，引力通量 E^i 与实 Ashtekar 联络 A^i 是体区域 $\sum$ 中的正则对。然而，如果坚持在体区域使用联络变量，我们会发现 ECH 拉格朗日量不会在孤立视界截面 H 上产生任何边界角项。若我们在体区域换回类矢量变量 (E^i, K^i) ，这类项就会出现；事实上，利用 Γ^i 的性质，可以证明式 (5) 可以改写为如下形式

$$\kappa \Theta_{\mathrm{ECH}} = \int_{\sum} E_i \wedge \delta K^i + \frac{1}{\gamma} \int_H e_i \wedge \delta e^i. \tag{6}$$

The implications of the corner symplectic potential in (6) in the quantum theory were studied in [44]. If we wish to keep the corner term and a connection variable in the bulk, we need to add a boundary Lagrangian term to L_{ECH} . This yields the Palatini Lagrangian $L_{\text{P}} = L_{\text{ECH}} + \text{d}\ell$, where the boundary Lagrangian reads [45].

(6) 中边界角辛势在量子理论中的影响已在文献 [44] 中研究。如果我们希望保留边界角项同时在体区保留联络变量，就需要向 L_{ECH} 添加一个边界拉格朗日项。由此得到帕拉蒂尼拉格朗日量 $L_{\text{P}} = L_{\text{ECH}} + \text{d}\ell$ ，其中边界拉格朗日量见 [45]。

$$\ell := \frac{1}{2\gamma} e_I \wedge \text{d}_\omega e^I. \tag{7}$$

The reason why we call it Palatini Lagrangian follows from the fact that the associated symplectic potential can be equivalently written as

我们将其称为帕拉蒂尼拉格朗日量的原因是，对应的辛势可以等价地写为

$$\kappa \Theta_{\text{P}} = \frac{1}{\gamma} \int_{\Sigma} E_i \wedge \delta A^i + \frac{1}{\gamma} \int_{H} e_i \wedge \delta e^i = \int_{\Sigma} E_i \wedge \delta K^i, \tag{8}$$

where in the second line we recognize the familiar canonical symplectic potential expressed in terms of Palatini variables.

其中我们可以在第二行中认出由帕拉蒂尼变量表示的常见正则辛势。

Constraints and Charges

约束与荷

The importance of comparing these two different formulations of gravity and their associated symplectic potentials stems from the fact that they lead to symmetry charges for the IH that can vanish or not according to which formulation one considers. This reflects the difference between gauge symmetries, which only label gauge redundancies and cannot be used to label physical states of, say, the IH - since by defining the corresponding Hamiltonian charges vanish on such states - and physical symmetries which, instead, possess non-vanishing charges. While this difference does not affect the standard BH entropy calculation in LQG, as reviewed below, it can have important implications for its interpretation regarding the nature of the DOF accounting for it and the role of the Immirzi parameter in the entropy counting (we will come back to this in section "Entropy DOF and the Immirzi Parameter").

比较这两种不同引力表述及其相关辛势的重要性源于: 它们给出的孤立视界 (IH) 对称性荷会因所采用的表述不同而可能非零或为零。这反映了规范对称性与物理对称性的区别: 规范对称性仅标记规范冗余，不能用来标记例如孤立视界的物理态——因为相应哈密顿荷在这类态上本来就为零；而物理对称性则拥有非零荷。如下文所述，尽管这一差异不影响圈量子引力 (LQG) 中标准黑洞熵的计算，但它会对熵对应的自由度 (DOF) 的本质、伊米尔齐参数在熵计数中的作用这些问题的解读产生重要影响 (我们会在“熵自由度与伊米尔齐参数”一节回到这一讨论)。

In the following we concentrate on static isolated horizons, namely we restrict ourselves to the non-rotating case (The staticity condition can be formulated by demanding the Newman-Penrose scalar component (in a null tetrad frame adapted to the IH geometry) $\mathrm{Im}\,(\Psi_2)$ to vanish [47].). As already pointed out, in order for the Hamiltonian time evolution to be well defined, the lapse at the horizon must vanish. Moreover, the time gauge fixes the internal boost freedom. Therefore, we only need to focus on SU(2) internal rotations with parameter α^i and spatial diffeomorphisms generated by tangent vector fields non-vanishing on the horizon $v \in T\left(\sum\right)$. In the bulk these transformations are generated by the Gauss constraint and the vector constraint, respectively,

下文我们聚焦静态孤立视界，即限定在非旋转情形 (静态条件可以表述为: 要求适配孤立视界几何的零标架中，纽曼-彭罗斯标量分量 $\mathrm{Im}\,(\Psi_2)$ 为零 [47])。如前所述，要使哈密顿时间演化良定义，视界处的时移必须为零。此外，时间规范固定了内部 boost 自由度。因此我们只需关注参数为 α^i 的 SU(2) 内部转动，以及由切向量场生成、在视界 $v \in T\left(\sum\right)$ 上非零的空间微分同胚。在体区中，这些变换分别由高斯约束和矢量约束生成,

$$\mathrm{d}_A E^i = 0,\ \left(v\lrcorner F^i\left(A\right)\right)\wedge E^i = 0. \tag{9}$$

By plugging the corresponding field transformations δ_α, δ_v into the symplectic form $\Omega = \delta\Theta$, on-shell of these constraints (denoted with $\doteq$), it is easy to show that [30]

将对应的场变换 δ_α, δ_v 代入辛形式 $\Omega = \delta\Theta$ ，在这些约束的壳上 (记为 $\doteq$)，可以很容易证明 [30]

$$\kappa\Omega_{\mathrm{P}}\left(\delta_\alpha, \delta\right) \triangleq 0,\ \kappa\Omega_{\mathrm{P}}\left(\delta_v, \delta\right) \triangleq \int_H \delta\left(\left(v\lrcorner E_i\right)\wedge K^i\right) = 0, \tag{10}$$

where the last equality holds due to the boundary conditions for static isolated horizons. Therefore, in the Palatini formulation defined by the symplectic potential (8), SU(2) rotations and tangent diffeomorphisms represent degenerate directions of the symplectic form for a static IH and their corresponding Hamiltonian charges vanish. Stated otherwise, in this case the IH symmetry group

其中最后一个等号成立源于静态孤立视界的边界条件。因此，在由辛势 (8) 定义的帕拉蒂尼表述中，SU(2) 转动和切向微分同胚是静态孤立视界辛形式的简并方向，它们对应的哈密顿荷为零。换言之，在这种情况下，孤立视界对称群

$$G_H = \mathrm{Diff}\left(H\right) \ltimes SU(2)^H \tag{11}$$

is trivially represented at the classical level for the static case.

在静态情形的经典层面是平凡表示。

On the other hand, if we consider the ECH formulation defined by the symplectic potential (5), one can show [45]

另一方面，如果我们考虑由辛势 (5) 定义的 ECH 表述，可以证明 [45]

$$\kappa\Omega_{\mathrm{ECH}}\left(\delta_\alpha,\delta\right) \triangleq \frac{1}{\gamma}\int_H \delta\left(\alpha^i E_i\right), \ \kappa\Omega_{\mathrm{ECH}}\left(\delta_v,\delta\right) \triangleq -\frac{2}{\gamma}\int_H \delta\left(v \lrcorner e_i \ \mathrm{d}e^i\right). \tag{12}$$

Hence, in this case the IH symmetry group (11) has a canonical representation in the gravitational phase space of an isolated horizon. In particular, it is straightforward to verify that the first set of charges in (12), $E[\alpha] = 1/(\kappa\gamma)\int_H \alpha^i E_i$, reproduces the non-commutativity relation of the LQG fluxes already at the classical and continuum level, as pointed out already in [30]. More precisely, by means of the corner symplectic potential expressing the non-commutativity of the frame field on H , we recover the $\mathfrak{su}(2)$ Lie algebra

因此，在这种情况下，孤立视界对称群 (11) 在孤立视界的引力相空间中存在正则表示。特别地，不难验证，(12) 中的第一组荷 $E[\alpha] = 1/(\kappa\gamma)\int_H \alpha^i E_i$ 早在经典连续层面就重现了圈量子引力通量的非对易关系，这一点在文献 [30] 中已经指出。更准确地说，借助表达 H 上标架场非对易性的边界角辛势，我们可以恢复 $\mathfrak{su}(2)$ 李代数

$$\{E[\alpha], E[\beta]\} = E[[\alpha,\beta]]. \tag{13}$$

These non-vanishing charges generate physical symmetries and can be used to label different geometrical states of the horizon. The infinite-dimensional IH symmetry group (11) is a subgroup of the corner symmetry group representing the universal maximally extended subgroup of bulk diffeomorphism in the presence of an embedded codimension-2 surface. A recent study of the symmetry group and charges of non-expanding horizons has been carried out in [48,49].

这些非零荷生成物理对称性，可以用来标记视界不同的几何态。无穷维孤立视界对称群 (11) 是边界角对称群的一个子群，边界角对称群是存在嵌入余维 2 曲面时，体微分同胚的通用极大延拓子群。近期文献 [48,49] 对非膨胀视界的对称群和荷开展了研究。

Finally, the passage to a connection parametrization of the IH corner phase space, which led to the original CS description of the boundary theory, can be achieved by means of a horizon constraint relating the curvature of the real Ashtekar connection at H to the corner flux. For spherical IHs considered in the rest of the chapter, this constraint takes the form

最后，要得到孤立视界边界相空间的联络参数化 (即原始边界理论的陈-西蒙斯描述)，可以利用一个视界约束，该约束将 H 处实阿西特卡联络的曲率与边界角通量联系起来。对于本章余下部分讨论的球形孤立视界，该约束形式为

$$F^i(A) \stackrel{H}{=} -\frac{\pi\left(1-\gamma^2\right)}{A_H} E^i. \tag{14}$$

Using this constraint, the symplectic form of spherical IHs in the Palatini formulation can be recast as [30]

利用该约束，帕拉蒂尼表述中球形孤立视界的辛形式可以改写为 [30]

$$\kappa\Omega_{\mathrm{P}}(\delta,\delta) = \frac{1}{\gamma}\int_\Sigma \delta E^i \wedge \delta A_i - \frac{A_H}{\pi\gamma\left(1-\gamma^2\right)}\int_H \delta A^i \wedge \delta A_i, \tag{15}$$

where the corner term now reproduces the symplectic structure of an SU(2) Chern-Simons theory at level

其中边界角项此时重现了 k 级 SU(2) 陈-西蒙斯理论的辛结构

$$k_{CS} = \frac{A_H}{4\pi\ell_P^2\gamma\,(1-\gamma^2)}. \tag{16}$$

The symplectic form (15) together with (14) represents the starting point for the original IH quantization yielding the single intertwiner model introduced below.

辛形式 (15) 结合约束 (14) 是原始孤立视界量子化的出发点，得到了下文介绍的单缠结模型。

As reviewed in section "History of Black Hole Entropy in LQG," the CS parametrization of the IH corner phase space was initially revealed in a U(1) gauge fixed formulation [17, 50]. The inclusion of distortion for static isolated horizons was introduced in both the SU (2) and U (1) constructions, respectively, in [31, 51] , while progress toward the inclusion of rotation was made in [32, 52– 55]. Alternative parametrizations involving BF-like variables and amenable to 2+1 LQG quantization techniques were proposed in [56-58]. A generalization to higher dimensional horizons and supersymmetry was formulated in [59-61]. The addition of gauge charges was studied originally in [16] and later more thoroughly in [62]. The extension to topologies different from the spherical one was considered in [63].

正如在“LQG 中黑洞熵的研究历史”一节中回顾的那样，孤立视界边界相空间的 Chern-Simons 参数化最初是在 U(1) 规范固定表述中提出的 [17, 50]。分别在 SU (2) 和 U (1) 构造中引入了静态孤立视界的畸变，相关工作收录于 [31, 51] ，而 [32, 52– 在纳入转动效应方面取得了进展 [55]。有人提出了使用类 BF 变量、适用于 2+1 维 LQG 量子化技术的替代参数化方案 [56-58]。也有人构建了适用于高维视界和超对称的推广形式 [59-61]。规范电荷的引入最早由文献 [16] 研究，之后文献 [62] 进行了更全面的研究。不同于球对称拓扑的推广由文献 [63] 讨论。

Quantum Geometry

量子几何

The symplectic structure (15) consists of a bulk and a corner contribution. The bulk term, parametrized by the real Ashtekar connection and its conjugate flux, lends itself to a quantization in terms of standard LQG techniques. This yields a bulk Hilbert space with an orthonormal basis of quantum geometry states spanned by spin networks. These are states labeled by a collection of links joining at nodes of arbitrary valence and forming a closed graph, a semi-integer positive number j (spin) - unitary irreducible representation of SU(2) - assigned to each link, and an invariant tensor (intertwiner) in the tensor product of SU(2) representations labeling the links converging at the given node. We refer the reader to the Chap. 83, "Emergence of Riemannian Quantum Geometry" for more details on the construction of the LQG Hilbert space of quantum geometry.

辛结构 (15) 由体贡献和边界角贡献组成。体项由实阿西卡联络及其共轭通量参数化，适合用标准圈量子引力 (LQG) 技术量子化，由此得到的体希尔伯特空间拥有由自旋网络张成的量子几何态正交归一基。这些态由一组要素标记: 任意价节点连接而成的闭合图，分配给每条链路的半正整数 j (自旋，即 SU(2) 的幺正不可约表示)，以及汇聚到给定节点的各链路标记 SU(2) 表示张量积中的不变张量 (缠结子)。关于 LQG 量子几何希尔伯特空间的构造细节，读者可参阅第 83 章“黎曼量子几何的涌现”。

In the presence of the spacetime inner boundary associated with the IH, some of the links can pierce the horizon 2-sphere. From an outside observer point of view, such links end at the horizon, where they create punctures labeled by both the link spin j and its corresponding magnetic number m . This way, the bulk Hilbert space can be represented as the orthogonal sum of open spin networks with one link ending at each of the n points on the horizon forming a given finite set P with $\{j\}_n = \{j_1, \dots, j_n\}$, namely

当存在与孤立视界 (IH) 关联的时空内边界时，部分链路可穿透视界二维球面。从外部观测者的视角看，这类链路终止于视界，并在该处留下由链路自旋 j 及其对应磁量子数 m 共同标记的穿孔。据此，体希尔伯特空间可表示为开自旋网络的正交和，其中每个链路终止于有限集 P 的 n 个视界点，满足 $\{j\}_n = \{j_1, \dots, j_n\}$ ，即

$$\mathcal{H}_{\Sigma} = \bigoplus_{P,\{j\}_n} \mathcal{H}_{\Sigma}^{P,\{j\}_n} \tag{17}$$

such that all states in each subspace $\mathcal{H}_{\Sigma}^{P,\{j\}_n}$ yield the same horizon area eigenvalue. This decomposition is useful for the entropy counting in the area ensemble (section ”Black Holes and Combinatorics”).

使得每个子空间 $\mathcal{H}_{\Sigma}^{P,\{j\}_n}$ 内的所有态都给出相同的视界面积本征值。该分解对面积系综的熵计数十分有用 (参见章节“黑洞与组合学”)。

The fact that the punctures are labeled also by (in general) non-vanishing magnetic numbers indicates that, in the quantum theory, local SU(2) charges defined on a small patch of the horizon around each puncture are different from zero. From the perspective of the Chern-Simons theory, these charges can be understood, by means of the IH constraints (14), as sources of conical curvature singularities. In order to restore local SU(2) gauge invariance at the punctures, one then needs to add corner DOF, which can be understood as topological defects sourcing a distributional curvature for the Chern-Simons theory. It is important to stress that, in this formulation, these so-called would-be-gauge DOF [64] have a purely quantum origin. This way, the corner Hilbert space can be identified with that of a Chern-Simons theory on a punctured 2-sphere with level k_{CS} given by (16), with flat curvature everywhere except at the location of the punctures in a given ordered set P [17, 29] . Upon identifying the Chern-Simons punctures DOF with the bulk spin labels, as a consequence of the quantum imposition of (14) - which restores local gauge invariance - the total Hilbert space in the presence of a quantum IH can be written as

穿孔也被 (通常) 非零的磁量子数标记，这一事实表明: 在量子理论中，定义在每个穿孔附近视界小区域上的局域 SU(2) 电荷不为零。从陈-西蒙斯理论的视角来看，通过孤立视界约束条件 (14)，这些电荷可被理解为锥形曲率奇点的源。为了恢复穿孔处的局域 SU(2) 规范不变性，我们需要引入边界角自由度，它可以被解释为拓扑缺陷，为陈-西蒙斯理论提供分布曲率源。需要强调的是，在该表述中，这些所谓的准规范自由度 [64] 有纯粹的量子起源。由此，边界角希尔伯特空间等价于能级为 (16) 给出的 k_{CS} 的穿孔二维球面上的陈-西蒙斯理论的希尔伯特空间，该理论中除有序集 P [17, 29] 指定的穿孔位置外处处曲率平坦。由于约束条件 (14) 的量子化 imposition 恢复了局域规范不变性，将陈-西蒙斯穿孔自由度与体自旋标记对应后，含量子孤立视界的总希尔伯特空间可写为

$$\mathcal{H}_{\mathrm{IH}} = \bigoplus_{P,\{j\}_n} \mathcal{H}_{\Sigma}^{P,\{j\}_n} \otimes \mathcal{H}_{H}^{P,\{j\}_n}. \tag{18}$$

Up to this point, the construction of the IH Hilbert space has been purely kinematical. In order to define the physical Hilbert space, spatial diffeomorphism invariance needs to be implemented as well (Recall that the Hamiltonian constraint does not play a direct role, as the lapse smearing function needs to vanish on the horizon.). This is a subtle but important aspect of the entropy counting. As pointed out at the beginning of section ”Constraints and Charges,” if the tangent diffeomorphism charges vanish at the horizon, as it does in the Palatini formulation (15), then the position of punctures on the horizon cannot be regarded as a physical quantity. This means that states corresponding to different localizations of the punctures on the horizon need to be considered as physically equivalent, and only the number n of punctures is required to characterize physical states. At the same time, in order for the quantization of the corner phase space to be well defined, an ordering of the punctures needs to be introduced. Moreover, for the entropy counting to yield a result linear in the horizon area, the punctures need to be considered distinguishable (See [65, 66] for the analysis and alternative models involving different statistics of the punctures.), so that different orderings count as different physical states and contribute to the entropy - the importance of distinguishability between two sets of punctures differing by their ordering was originally pointed out by Krasnov [67]. Different orderings can be obtained by the action of tangent diffeomorphisms, which act transitively on this additional structure. This means that, in the quantum theory, a subset of diffeomorphism charges at the horizon needs to be activated, the same way as a finite set of local SU(2) charges are. In the standard treatment then, also these extra and necessary diffeo DOF are considered to have a purely quantum origin - we will come back to this important interpretational aspect in section ”Entropy DOF and the Immirzi Parameter”.

到目前为止，孤立视界 (IH) 希尔伯特空间的构造都只是运动学层面的。要定义物理希尔伯特空间，还需要实现空间微分同胚不变性 (回想一下，哈密顿约束并不直接发挥作用，因为推移弥散函数在视界上必须为零)。这是熵计数中微妙但重要的一环。正如“约束与电荷”一节开头指出的，如果切向微分同胚电荷在视界上为零——就像帕拉蒂尼表述 (15) 中的情况那样，那么刺点在视界上的位置就不能被视为物理量。这意味着，对应刺点在视界上不同位置的态需要被看作物理上等价的，只有刺点的数量 n 是刻画物理态所必需的。同时，为了让拐角相空间的量子化良定义，需要给刺点引入一个排序。此外，要让熵计数得到与视界面积成正比的结果，刺点需要被视为可区分的 (关于分析和涉及刺点不同统计的替代模型见 [65, 66])，因此不同的排序会被算作不同的物理态，对熵有贡献——刺点排序不同的两个集合可区分的重要性最初由 Krasnov 指出 [67]。不同排序可以通过切向微分同胚的作用得到，切向微分同胚在这个额外结构上可递作用。这意味着，在量子理论中，需要激活视界处微分同胚电荷的一个子集，就像局部 SU(2) 电荷的有限集合那样。在标准处理中，这些额外且必要的微分同胚自由度也被认为具有纯粹的量子起源——我们会在“熵自由度与伊米尔齐参数”一节回到这个重要的诠释问题。

Hence, upon imposition of the spatial diffeomorphism constraint and keeping in mind the distinguishable statistical character of the punctures, the IH physical Hilbert space for given number n of punctures can be written as

因此，在施加空间微分同胚约束并牢记刺点的可区分统计性质后，给定刺点数量 n 的孤立视界物理希尔伯特空间可以写为

$$\mathcal{H}_{\mathrm{IH}}^{n}=\bigoplus_{\{j\}_n}\mathcal{H}_{\Sigma}^{\{j\}_n}\otimes\mathcal{H}_{H}^{CS}(j_1,\ldots,j_n), \tag{19}$$

where the spins $j_1,\ldots,j_n$ are subject to the horizon area constraint

其中自旋 $j_1,\ldots,j_n$ 满足视界面积约束

$$A_H-\delta\leq 4\pi\gamma\ell_P^2\sum_{i=1}^{n}\sqrt{j_i(j_i+1)}\leq A_H+\delta, \tag{20}$$

where we introduced an area interval with δ of the order of the Planck area. Finally, there is an additional global constraint that follows from (14) and the spherical topology of the horizon, which implies that a loop going around all the punctures is contractible and, hence, the holonomy along it should be trivial. This implies the inclusion

这里我们引入了一个面积区间，其 δ 为普朗克面积量级。最后，由 (14) 和视界的球面拓扑可以得到一个额外的整体约束: 环绕所有刺点的圈是可缩的，因此沿该圈的和乐应当是平凡的。这说明需要包含

$$\mathcal{H}_{H}^{CS}(j_1\cdots j_n)\subset\mathrm{Inv}(j_1\otimes\cdots\otimes j_n), \tag{21}$$

where Inv denotes the invariant subpace in the tensor product, which becomes an equality in the large area limit $A_H\propto k_{CS}\rightarrow\infty$. Therefore, for a large BH, the horizon Hilbert space can be effectively identified with an SU(2) intertwiner space between the spins associated with all the punctures. This single intertwiner picture can also be understood as the result of a coarse graining procedure of the DOF of a spin network in the BH interior [68].

其中 Inv 表示张量积中的不变子空间，在大面积极限 $A_H\propto k_{CS}\rightarrow\infty$ 下该式变为等式。因此，对于大黑洞，视界希尔伯特空间可以被有效地等同于所有刺点关联自旋之间的 SU(2) 交缠子空间。这个单交缠子图像也可以理解为对黑洞内部自旋网络自由度做粗粒化处理的结果 [68]。

By further tracing over the exterior bulk DOF while restricting to horizon states compatible with (20), in accordance with the weak holographic principle advocated above, we arrive at the IH density matrix ρ_{IH} . Demanding the final BH state to be a maximally mixed state or equivalently the validity of a maximal entropy principle - which is expected to capture some relevant features of the effective dynamics in the continuum limit [69], the quantum statistical mechanical horizon entropy is given by

根据我们前文提倡的弱全息原理，进一步对外部体自由度取迹，同时限制到满足 (20) 的视界态，我们就得到了孤立视界密度矩阵 ρ_{IH} 。要求最终黑洞态是最大混合态，等价于要求最大熵原理成立——该原理被认为可以抓住连续极限中有效动力学的一些相关特征，因此量子统计力学的视界熵由下式给出

$$S = -\operatorname{Tr}(\rho_{\mathrm{IH}} \ln \rho_{\mathrm{IH}}) = \ln(\mathcal{N}_{\mathrm{IH}}), \tag{22}$$

where $\mathcal{N}_{\mathrm{IH}}$ is the dimension of the IH corner Hilbert space. Its derivation through combinatorial methods is the subject of the next section.

其中 $\mathcal{N}_{\mathrm{IH}}$ 是孤立视界拐角希尔伯特空间的维数。通过组合方法推导它是下一节的主题。

Black Holes and Combinatorics

黑洞与组合数学

In order to compute the entropy of a physical system one has to count the number of microscopic configurations compatible with its macroscopic state, i.e., solve a combinatorics problem. Ideally, one would like to have closed expressions for the entropy, but often, it is necessary to work with asymptotic expansions to really understand the behavior of the entropy in the large area limit (for which we expect that the ”static” description discussed above is good enough).

要计算物理系统的熵，必须统计与该系统宏观状态相容的微观构型数量，即求解一个组合数学问题。理想情况下我们希望得到熵的闭合表达式，但通常我们需要使用渐近展开才能真正理解大面积极限下熵的行为，在该极限下我们预期前文讨论的「静态」描述足够准确。

The specific nature of the combinatorial problems relevant to the computation of BH entropy makes it necessary to work with diophantine equations. Two other technical tools are also useful: generating functions and Laplace transforms (as first pointed out in [19]). As we will show, from the Laplace transform of the BH entropy as a function of its area, it is possible to obtain the large area behavior. This is how the Hawking law is recovered in this setting. In the context of LQG, the use of combinatorial methods shows up at a very basic level, for instance, when trying to understand the area operator. The distribution of area eigenvalues can be studied in great detail [70] by using methods similar to those employed to compute the entropy, so we will give a short account on this problem in the following.

黑洞熵计算所对应的组合问题具有特殊性，需要使用丢番图方程进行研究。另外还有两项常用技术工具: 生成函数与拉普拉斯变换 (文献 [19] 最早指出这一点)。正如我们将要展示的，通过黑洞熵关于其面积的拉普拉斯变换，可以得到大面积极限下的行为，霍金定律正是在这套框架下得到恢复的。在圈量子引力 (LQG) 中，组合方法的应用出现在非常基础的层面，例如在研究面积算符时就会用到。利用和计算熵时相似的方法，可以非常细致地研究面积本征值的分布 [70]，因此我们在下文中会对该问题做简要说明。

The Spectrum of the Area Operator

面积算符的谱

The spectrum of the area operator $\widehat{A}_S$ associated with a surface S has a complicated structure (see [71, 72]), and however, when studying BHs in LQG modelled with the help of isolated horizons, the relevant part of it is given by

与曲面 S 关联的面积算符 $\widehat{A}_S$ 的谱结构复杂 (见文献 [71, 72])，但在研究借助孤立视界建模的圈量子引力黑洞时，谱的相关部分可表示为

$$A_s = 4\pi\gamma\ell_P^2 \sum_{j=1}^{n} \sqrt{k_j\left(k_j+2\right)};\ k_j \in \mathbb{N};\ n \in \mathbb{N}. \tag{23}$$

For simplicity, in the following we will use units such that $4\pi\gamma\ell_P^2 = 1$. The combinatorial problem that must be solved in order to describe the distribution of the area eigenvalues can be phrased as follows:

为简化表述，下文我们采用满足 $4\pi\gamma\ell_P^2 = 1$ 的单位制。描述面积本征值分布需要解决的组合问题可表述如下:

For every positive number $a > 0$, determine the function $N(a)$ defined as one plus the number of different multisets consisting of positive integers $k_j \in \mathbb{N}$ such that

对任意正数 $a > 0$ ，确定函数 $N(a)$ ，该函数定义为 1 加上满足条件的、由正整数 $k_j \in \mathbb{N}$ 构成的不同多重集的个数，其中条件为

$$\sum_{j\in\mathbb{N}} \sqrt{k_j\left(k_j+2\right)} \leq a.$$

The value of $N(a)$ tells us the number of eigenvalues of the area in the interval $[0, a]$ taking into account the degeneracy associated with the fact that different multisets of positive integers may give the same area eigenvalue. Several approximate ways to study this problem have been discussed in the literature (see [70] and the references therein), and however, it is possible to tackle it without relying on popular, but difficult-to-control, approximations such as $\sqrt{k_j\left(k_j+2\right)} \sim k_j$ and $\sqrt{k_j\left(k_j+2\right)} \sim k_j+1$. Notice, by the way, that $N(a)$ is a staircase function, i.e., an increasing function that is constant except at a countable set of values of the independent variable a where it jumps (precisely the eigenvalues of the area operator). The Laplace transform (If $f : \mathbb{R}^+ \to \mathbb{R} : x \mapsto f(x)$, we denote its

$N(a)$ 的值给出了区间 $[0, a]$ 内面积本征值的总数，同时计入了简并度——不同的正整数多重集可能给出相同的面积本征值。文献中已经讨论了该问题的多种近似研究方法 (见 [70] 及其中参考文献)，但我们可以不依赖诸如 $\sqrt{k_j\left(k_j+2\right)} \sim k_j$ 和 $\sqrt{k_j\left(k_j+2\right)} \sim k_j+1$ 这类常用却难以控制的近似来处理这个问题。顺便说明，$N(a)$ 是阶梯函数，即它是增函数，除了在可数个自变量 a 处发生跳跃外保持不变，跳跃点恰好就是面积算符的本征值。拉普拉斯变换 (若 $f : \mathbb{R}^+ \to \mathbb{R} : x \mapsto f(x)$ ，我们将其

Laplace transform as $\mathcal{L}(f;s) := \int_0^\infty e^{-sx} f(x)\,\mathrm{d}x$.) of functions of this type can often be obtained in closed form because the Heaviside $\theta(a-a_0)$ step function $(a_0 > 0)$ can be expressed as (The integration contour, denoted with the limits $c - i\infty$ and $c + i\infty$, is the straight line, parallel to the imaginary axis, $\mathrm{Re}(z) = c$ with c larger than the real part of the singularity in the integrand.)

拉普拉斯变换记为 $\mathcal{L}(f;s) := \int_0^\infty e^{-sx} f(x)\,\mathrm{d}x$ 。) 这类函数的拉普拉斯变换通常可以得到闭合形式，因为海维赛德 $\theta(a-a_0)$ 阶跃函数 $(a_0 > 0)$ 可以表示为 (积分围道由限界 $c - i\infty$ 和 $c + i\infty$ 标记，是平行于虚轴的直线，位置为 $\mathrm{Re}(z) = c$ ，且 c 大于被积函数奇点的实部。)

$$\theta(a-a_0) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{(a-a_0)s}}{s}\,\mathrm{d}s,\ (c > 0),$$

and, hence, if the jumps of $N(a)$ at $a_n > 0\,(n \in \mathbb{N})$ are β_n , we can write

因此，若 $N(a)$ 在 $a_n > 0\,(n \in \mathbb{N})$ 处的跳跃量为 β_n ，我们可以写出

$$N(a) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{as}}{s}\widehat{N}(s),\ \text{ with }\ \widehat{N}(s) := \sum_n \beta_n e^{-a_n s}. \tag{24}$$

In practice, the effectiveness of this strategy hinges on the possibility to write the function $\widehat{N}(s)$ in an appropriate closed form. As discussed in [73,74], by using generating functions and some properties of the solutions to the Pell equation, it is actually possible to find the following expression for $\widehat{N}(s)$:

实际上，该方法的有效性取决于能否将函数 $\widehat{N}(s)$ 写成合适的闭合形式。正如文献 [73,74] 中讨论的，利用生成函数和佩尔方程解的若干性质，我们确实可以得到 $\widehat{N}(s)$ 的下述表达式:

$$\widehat{N}(s) = \prod_{k=1}^{\infty} \frac{1}{1 - \exp\left(-s\sqrt{k(k+2)}\right)}. \tag{25}$$

Plugging this into (24) provides an integral representation for $N(a)$ which can be used to extract the asymptotic behavior of $N(a)$ when $a \to \infty$ and approximate expressions in other regimes (for instance, for small values of a) [70].

将其代入式 (24) 即可得到 $N(a)$ 的积分表示，利用该积分表示可以得到 $a \to \infty$ 趋于无穷时 $N(a)$ 的渐近行为，还可以得到其他区域 (例如 a 取小值时) 的近似表达式 [70]。

As the derivation of (25) is actually the first step in the BH entropy computations, we will give now some details about it. An important preliminary comment is that the eigenvalues of the area (23) can always be written as linear combinations, with integer coefficients, of square roots of square-free numbers (These are positive integers that can be written as products of different primes, i.e., such that their prime number decomposition has no repeated factors.) $p_j\,(j \in \mathbb{N}), (p_1 = 2, p_2 = 3$, $p_3 = 5, p_4 = 6, \ldots)$. If a certain linear combination $\tilde{a} := \sum_{k=1}^{r} q_k\sqrt{p_k}$ is to be an eigenvalue of the area operator, it must be possible to find solutions to the equation

由于式 (25) 的推导实际上是黑洞熵计算的第一步，我们现在将对此给出更多细节。一个重要的预备说明是: 面积 (23) 的本征值总可以写为无平方因子数平方根的整系数线性组合 (无平方因子数是可写作不同质数乘积的正整数，即其质因数分解中没有重复因子) $p_j\,(j\in\mathbb{N}),(p_1=2,p_2=3$ ，$p_3=5,p_4=6,\ldots)$ 。若某个线性组合 $\tilde{a}\ :=\ \sum_{k=1}^{r} q_k\sqrt{p_k}$ 要成为面积算符的本征值，必须能找到下述方程的解

$$\sum_{k\in\mathbb{N}} n_k\sqrt{k\,(k+2)}=\sum_{j=1}^{r} q_j\sqrt{p_j} \tag{26}$$

in the unknowns $k\in\mathbb{N}$ and $n_k\in\mathbb{N}$. A solution (k,n_k) means that a state with n_k edges piercing the horizon and carrying $k/2$ spin labels is an eigenfunction of the area operator with eigenvalue $\tilde{a}$. If several solutions exist, the eigenvalue is degenerate, and if no solution exists, then $\tilde{a}$ does not belong to the spectrum of the area operator. Each term of the form $\sqrt{k\,(k+2)}$ in (26) can be written as the product of an integer times the square root of a square-free number (SRSFN), and hence, the left-hand side of (26) will always be a linear combination of SRSFN's with coefficients given by integer linear combinations of the n_k . Then, we have to identify, for each of the p_j appearing in $\tilde{a}$, the possible values of k satisfying $\sqrt{k\,(k+2)}=\sqrt{(k+1)^2-1}=y\sqrt{p_j}$ for some $y\in\mathbb{N}$. This amounts to solving the Pell equation

关于未知量 $k\in\mathbb{N}$ 和 $n_k\in\mathbb{N}$ 。解 (k,n_k) 意味着: 存在一个状态，有 n_k 条边穿透视界并携带 $k/2$ 自旋标签, 它是面积算符对应本征值 $\tilde{a}$ 的本征函数。若存在多个解, 则该本征值是简并的; 若不存在解, 则 $\tilde{a}$ 不属于面积算符的谱。式 (26) 中每个 $\sqrt{k\,(k+2)}$ 形式的项都可以写为一个整数乘以一个无平方因子数平方根 (SRSFN) 的乘积，因此式 (26) 的左侧始终是 SRSFN 的线性组合，其系数为 n_k 的整数线性组合。接下来, 对于出现在 $\tilde{a}$ 中的每个 p_j , 我们需要确定满足 $\sqrt{k\,(k+2)}=\sqrt{(k+1)^2-1}=y\sqrt{p_j}$ (对某个 $y\in\mathbb{N}$ 成立) 的 k 的可能取值。这等价于求解佩尔方程

$$(k+1)^2-p_j y^2=1 \tag{27}$$

in the unknowns k and y for each p_j . The solutions to (27) are well known. For each square-free p_j , they are an infinite set of the form $\left(k_m^j,y_m^j\right)$ labeled by $m\in\mathbb{N}$. They all derive from a fundamental solution $\left(k_1^i,y_1^i\right)$ - which corresponds to the lowest values of k and y - and are given by a simple formula (see [23]). For instance, for $p_1=2$ the fundamental solution to the Pell equation (27) is $(2,2)$ and the first solutions are $(2,2),(16,12),(98,70),\ldots$

对每个 p_j , 关于未知量 k 和 y 求解。式 (27) 的解是已知的。对每个无平方因子 p_j , 解是形如 $\left(k_m^j,y_m^j\right)$ 的无穷集合，由 $m\in\mathbb{N}$ 标记。所有解都可由基本解 $\left(k_1^i,y_1^i\right)$ 导出——基本解对应 k 和 y 的最小值——并可通过简单公式给出 (参见 [23])。例如，对 $p_1=2$ ，佩尔方程 (27) 的基本解是 $(2,2)$ ，前几个解为 $(2,2),(16,12),(98,70),\ldots$

By doing this for all the square-free numbers p_i appearing in $\tilde{a}$, we can put the left-hand side of (26) in the form $\sum_{j=1}^{r}\sum_{m=1}^{\infty} n_{k_m^j}\,y_m^j\sqrt{p_j}$, with $n_{k_m^j}$ nonnegative integers, and write (26) as

对出现在 $\tilde{a}$ 中的所有无平方因子数 p_i 完成上述步骤后，我们可以将式 (26) 左侧整理为 $\sum_{j=1}^{r}\sum_{m=1}^{\infty} n_{k_m^j} y_m^j \sqrt{p_j}$ 的形式，其中 $n_{k_m^j}$ 为非负整数，进而将式 (26) 写为

$$\sum_{j=1}^{r}\sum_{m=1}^{\infty} n_{k_m^j} y_m^j \sqrt{p_j} = \sum_{j=1}^{r} q_j \sqrt{p_j}. \tag{28}$$

Taking into account that the $\sqrt{p_j}$ are linearly independent over the rationals, equation (28) can actually be written as the system of r equations

考虑到 $\sqrt{p_j}$ 在有理数域上线性无关，方程 (28) 实际上可以改写为由 r 个方程组成的方程组

$$\sum_{m=1}^{\infty} y_m^j n_{k_m^j} = q_j, \ \ j = 1, \dots, r \tag{29}$$

in the unknowns $n_{k_m^j}$. It should be noted that the sum in (29) is always finite with a number of terms that depends on q_j (the y_m^j grow with m). Another important fact is that, for different square-free numbers p_{j_1} and p_{j_2} , the sets $\left\{k_m^{j_1} : m \in \mathbb{N}\right\}$ and $\left\{k_m^{j_2} : m \in \mathbb{N}\right\}$ are always disjoint. This can be shown by noting that if $k \in \left\{k_m^{j_1} : m \in \mathbb{N}\right\} \cap \left\{k_m^{j_2} : m \in \mathbb{N}\right\}$, then there exist positive integers y_1 and y_2 such that $p_1/p_2 = y_1^2/y_2^2$, but this is impossible. Indeed, the irreducible form of the fraction p_1/p_2 is the quotient of two square-free numbers (say, π_1 and π_2), and the irreducible form of the fraction y_1^2/y_2^2 is the quotient of the squares of two integers (say, z_1 and z_2). Now, the irreducible form of a fraction is unique, hence, $\pi_1 = z_1^2$ and $\pi_2 = z_2^2$, which leads to a contradiction as π_1 and π_2 are square-free.

针对未知量 $n_{k_m^j}$ 。需要注意的是，(29) 中的求和总是有限的，项数取决于 q_j (y_m^j 随 m 增大)。另一个重要结论是: 对于不同的无平方因子数 p_{j_1} 和 p_{j_2} ，集合 $\left\{k_m^{j_1} : m \in \mathbb{N}\right\}$ 与 $\left\{k_m^{j_2} : m \in \mathbb{N}\right\}$ 始终不相交。这可以通过下述方式证明: 若 $k \in \left\{k_m^{j_1} : m \in \mathbb{N}\right\} \cap \left\{k_m^{j_2} : m \in \mathbb{N}\right\}$，则存在正整数 y_1 和 y_2 满足 $p_1/p_2 = y_1^2/y_2^2$ ，但这是不可能的。事实上，分数 p_1/p_2 的既约形式是两个无平方因子数的商 (记为 π_1 和 π_2)，而分数 y_1^2/y_2^2 的既约形式是两个整数平方的商 (记为 z_1 和 z_2)。由于分数的既约形式唯一，因此可得 $\pi_1 = z_1^2$ 和 $\pi_2 = z_2^2$ ，但 π_1 和 π_2 都是无平方因子数，由此推出矛盾。

We then see that the variables $n_{k_m^j}$ appearing in each of the equations in (29) are different. As a consequence, the equations are independent and can be solved separately. There are, in fact, well-known algorithms to do this. However, for our purposes it suffices to know the number of solutions to these equations. This can be easily achieved by using generating functions. For instance, let us consider the following equation in the nonnegative, integer unknowns $z_j, j \in \mathbb{N}$ (here, $a_j \in \mathbb{N}$):

由此可知，(29) 中每个方程出现的变量 $n_{k_m^j}$ 都是不同的。因此这些方程相互独立，可以分别求解。事实上，已有成熟算法可以完成这项工作。但对我们的研究而言，只需要知道这些方程的解的个数即可，这可以通过生成函数轻松得到。例如，考虑下述关于非负整数未知量 $z_j, j \in \mathbb{N}$ 的方程 (此处 $a_j \in \mathbb{N}$):

$$\sum_{j=1}^{N} a_j z_j = q$$

for each $q \in \mathbb{N}$. Now, the number of solutions to this equation is the coefficient of the x^q term in the Taylor expansion about $x = 0$ of the function

对每个 $q \in \mathbb{N}$ 成立。此时该方程的解的个数就是函数在 $x = 0$ 处泰勒展开中 x^q 项的系数

$$f(x) = \frac{1}{\prod_{j=1}^{N} (1 - x^{a_j})},$$

which we denote as $[x^q] f(x)$ - as can be easily seen by multiplying the Taylor expansions of $1/(1 - x^{a_j})$, and tracking the origin of the terms that add up to give each power of x in the expansion. By proceeding in this way, it is straightforward to see that the number of solutions to (28) can be written as $\left[x_1^{q_1} \cdots x_r^{q_r}\right] G(x_1, x_2, \ldots)$ with G given by the following formal series involving an infinite number of variables x_j , each one of them associated with the corresponding square-free integer p_j :

我们将该系数记为 $[x^q] f(x)$ ——只需将 $1/(1 - x^{a_j})$ 的泰勒展开相乘，追踪展开中得到 x 各次幂的项的来源即可轻松得到这一结论。按照这种思路，不难发现 (28) 的解数可以写为 $\left[x_1^{q_1} \cdots x_r^{q_r}\right] G(x_1, x_2, \ldots)$ ，其中 G 由下述包含无穷多个变量 x_j 的形式级数给出，每个变量对应一个无平方因子整数 p_j :

$$G(x_1, x_2, \ldots) = \prod_{j=1}^{\infty} \prod_{m=1}^{\infty} \frac{1}{1 - x_j^{y_m^j}}. \tag{30}$$

Notice that, for a concrete choice of a finite number of q_j , one only needs to consider a truncation of G involving a finite number of variables - the ones associated with the square-free numbers appearing in the right-hand side of (26). Also, given a particular value of $\tilde{a}$, the maximum value of each y_m^j is bounded by $\tilde{a}$; this constraints the possible values of m . Now, we are in the position to finally obtain (25). In order to do so, we simply have to turn the terms of the form $x_1^{q_1} \cdots x_r^{q_r}$ in the power series expansion of G into $\exp\left(-s \sum_{j=1}^{r} q_j \sqrt{p_j}\right)$. This can easily be achieved by replacing each x_j by $\exp\left(-s\sqrt{p_j}\right)$. This leads to

请注意, 对于具体选定的有限个 q_j , 我们只需考虑截断后的 G , 它仅包含有限个变量——即与式 (26) 右侧出现的无平方因子数相关联的变量。此外，给定 $\tilde{a}$ 的特定值后，每个 y_m^j 的最大值都受 $\tilde{a}$ 约束；这也限制了 m 的可能取值。现在我们终于可以推导出式 (25) 了。为此，我们只需将 G 幂级数展开中形如 $x_1^{q_1} \cdots x_r^{q_r}$ 的项替换为 $\exp\left(-s \sum_{j=1}^{r} q_j \sqrt{p_j}\right)$ ，这可以很容易地通过将每个 x_j 替换为 $\exp\left(-s\sqrt{p_j}\right)$ 实现，由此得到:

$$\widehat{N}(s) = G\left(e^{-s\sqrt{p_1}}, e^{-s\sqrt{p_2}}, \ldots\right) = \prod_{j=1}^{\infty} \prod_{m=1}^{\infty} \frac{1}{1 - e^{-s y_m^j \sqrt{p_j}}} = \prod_{j=1}^{\infty} \prod_{m=1}^{\infty} \frac{1}{1 - e^{-s\sqrt{k_m^j (k_m^j + 2)}}}.$$

To end, we notice that, as $\left\{k_m^{j_1} : m \in \mathbb{N}\right\}$ and $\left\{k_m^{j_2} : m \in \mathbb{N}\right\}$ are always disjoint and every $k \in \mathbb{N}$ is the solution to some Pell equation (corresponding to a square-free number that can be identified by computing $\sqrt{k(k+2)}$ and taking out from the square root as many factors as possible), the products appearing in the preceding expression can be written as in (25).

最后我们注意到，由于 $\{k_m^{j_1}:m\in\mathbb{N}\}$ 和 $\{k_m^{j_2}:m\in\mathbb{N}\}$ 始终不相交，且每个 $k\in\mathbb{N}$ 都是某个佩尔方程的解 (对应一个无平方因子数，可通过计算 $\sqrt{k(k+2)}$ 并从平方根中提出尽可能多的因子来确定)，前述表达式中的乘积可以写为式 (25) 的形式。

Several comments are in order now:

现在我们给出几点说明:

- The expression for $N(a)$ given by (24,25) is very well suited to analyze the spectrum of the area operator (23) because it encodes both the position of the area eigenvalues and the degeneracy associated with the fact that the same area eigenvalue can correspond to different spin network states.

- 式 (24)(25) 给出的 $N(a)$ 表达式非常适合分析面积算符 (23) 的谱，因为它同时编码了面积本征值的位置，以及同一面积本征值可对应不同自旋网络态带来的简并度。

- It should be pointed out, however, that the number of area eigenstates smaller than or equal to a given area a is given by $\lim_{A\to a^+} N(a)$ and not by $N(a)$. This is so because the value of the inverse Laplace transform at a jump singularity is the average of the left and right limits there.

- 但需要指出，不大于给定面积 a 的面积本征态数量由 $\lim_{A\to a^+} N(a)$ 给出，而非 $N(a)$。这是因为逆拉普拉斯变换在跳跃奇点处的值是该点左右极限的平均值。

- It is important to realize that if the area spectrum were equally spaced, it would be possible to encode the content of the function $N(a)$ in a (formal) power series of a single variable. In this sense, Laplace transforms prove to be far superior because they can accommodate more general situations as the one relevant here.

- 需要认识到: 若面积谱是等间距的，那么单变量形式幂级数就足以承载函数 $N(a)$ 的内容。就此而言，拉普拉斯变换的优势要大得多，因为它可以适配本文涉及的这类更一般的情况。

Black Hole Entropy Computations

黑洞熵计算

In the preceding subsection we have looked at the spectrum of the area operator. We will briefly explain now how the BH entropy is defined according to some prescriptions considered in the literature (We will not be exhaustive here, and we will just consider a particular example which, nonetheless, we consider as sufficiently illustrative.) and how it can be computed by using generating functions along the lines spelled above. Here we will focus on the $U(1)$ case; we will make some comments on the $SU(2)$ case at the end of the section. A typical phrasing of the counting problem that must be solved in order to compute the entropy of a BH is the following [18]:

在前一小节中，我们已经研究了面积算符的谱。现在我们将简要说明，如何按照文献中已有的规则定义黑洞熵 (我们在此不作穷举，仅讨论一个代表性的具体例子)，以及如何按照前文梳理的思路利用生成函数计算黑洞熵。本文我们将聚焦 $U(1)$ 的情况，在本节末尾对 $SU(2)$ 的情况作补充讨论。为计算黑洞熵，需要解决的计数问题的典型表述如下 [18]:

The entropy $S(a)$ of a BH of area a is $\log(1+N(a))$, where $N(a)$ is the number of all the arbitrarily long, finite, sequences $(k_1,\ldots,k_n)$ of nonzero integers such that the following two conditions hold:

面积为 a 的 BH 的熵为 $S(a)$ ，即 $\log(1+N(a))$ ，其中 $N(a)$ 是满足以下两个条件的所有任意长有限非零整数序列 $(k_1,\ldots,k_n)$ 的数量:

$$\sum_{j=1}^{N}\sqrt{|k_j|(|k_j|+2)}\leq a,\ \sum_{j=1}^{N}k_j=0.$$

The problem can be solved by following these four steps:

该问题可以通过以下四个步骤求解:

1. For a given value $\tilde{a}$ of the area, find the number of ways to choose positive integers $|k_j|\in\mathbb{N}$ such that

1. 对给定的面积值 $\tilde{a}$ ，求出满足条件的正整数 $|k_j|\in\mathbb{N}$ 的选取方式数:

$$\sum_{j=1}^{N}\sqrt{|k_j|(|k_j|+2)}=\tilde{a}. \tag{31}$$

At this stage we do not care about order, i.e., we only need to find out how many times each integer appears (we just count multisets).

在这一步我们不考虑顺序，即只需要统计每个整数出现的次数 (我们仅对多重集计数)。

2. Count the possible ways of ordering the multisets obtained in step 1.

2. 对步骤 1 得到的多重集，统计所有可能的排列方式数。

3. Count all the ways to introduce signs in the sequences of integers considered in step 2 in such a way that the condition $\sum_{j=1}^{N}k_j=0$ holds. Here k_j refers to $|k_j|$ with a positive or a negative sign.

3. 对步骤 2 得到的整数序列，统计所有能引入符号且满足条件 $\sum_{j=1}^{N}k_j=0$ 的方式数。此处 k_j 指带正号或负号的 $|k_j|$ 。

4. Repeat for all the area eigenvalues smaller than or equal to $\tilde{a}$ and add the results.

4. 对所有小于等于 $\tilde{a}$ 的面积本征值重复上述步骤，将结果求和。

Step 1 has been essentially solved in section ”The Spectrum of the Area Operator” where we discussed how to count the number of solutions to the diophantine equations that tell us the different ways to get a given area eigenvalue. As we showed, a neat way to encode this information was to use the generating function (30). In this step we just determine the number of multisets (configurations) $\left\{\left(k_m^j, n_{k_m^j}\right)\right\}$ associated with a given $\tilde{a} = \sum_j q_j \sqrt{p_j}$.

步骤 1 实质上已在「面积算符的谱」一节中解决，我们在该节讨论了如何对得到给定面积本征值的不同方式对应的丢番图方程的解计数。正如我们所示，利用生成函数 (30) 可以简洁地编码该信息。在这一步我们只需确定与给定 $\tilde{a} = \sum_j q_j \sqrt{p_j}$ 关联的多重集 (构型) $\left\{\left(k_m^j, n_{k_m^j}\right)\right\}$ 的数目。

In Step 2, let us consider a configuration

在步骤 2 中，我们考察一个构型

$$\left(\underbrace{1, \ldots, 1}_{n_1}, \ldots, \underbrace{k, \ldots, k}_{n_k}, \ldots, \underbrace{k_{\max}, \ldots, k_{\max}}_{n_{k_{\max}}}\right), \; n_1, n_2, \ldots, n_{k_{\max}} > 0.$$

The number of ways to reorder the elements in this configuration is just given by the multinomial coefficient $\left(\sum_{k=1}^{k_{\max}} n_k\right)! / \prod_{k=1}^{k_{\max}} n_k!$.

该构型中元素重排的方式数恰好由多项式系数 $\left(\sum_{k=1}^{k_{\max}} n_k\right)! / \prod_{k=1}^{k_{\max}} n_k!$ 给出。

We can now modify the generating function (30) in such a way that the coefficient of each term gives us the number of possible reorderings. The way to do this is explained in [74]. The final result is

现在我们可以修改生成函数 (30)，使得每一项的系数给出所有可能的重排数。该方法详见文献 [74]，最终结果为

$$G^{(2)}(x_1, x_2, \ldots) = \left(1 - \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} x_j^{y_m^j}\right)^{-1}. \tag{32}$$

Step 3 The condition $\sum_{j=1}^{N} k_j = 0$ (often referred to as the projection constraint) can be taken into account by including an extra variable in (32). As discussed in [74,75], the generating function

步骤 3 条件 $\sum_{j=1}^{N} k_j = 0$ (通常称为投影约束) 可以通过在 (32) 中引入一个额外变量来纳入考虑。正如文献 [74,75] 所讨论的，生成函数

$$G^{\mathrm{DL}}(z, x_1, x_2, \ldots) = \frac{1}{1 - \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} \left(z^{k_m^j} + z^{-k_m^j}\right) x_j^{y_m^j}} \tag{33}$$

(here the label DL stands for Domagala-Lewandowski) is such that

(此处标记 DL 代表 Domagala-Lewandowski) 满足

$$\left[x_1^{q_1} x_2^{q_2} \cdots\right]\left[z^0\right] G^{\mathrm{DL}}(z, x_1, x_2, \ldots) =: \left[x_1^{q_1} x_2^{q_2} \cdots\right] \widetilde{G}^{\mathrm{DL}}(x_1, x_2, \ldots)$$

gives the number of sequences of integers $(k_1, \ldots, k_N)$ satisfying the conditions

给出满足条件的整数序列 $(k_1, \ldots, k_N)$ 的数目

$$\sum_{j=1}^{N} \sqrt{|k_j|\left(|k_j|+2\right)} = a, \ \sum_{j=1}^{N} k_j = 0.$$

We will write this number as $D^{\mathrm{DL}}\left(q_1\sqrt{p_1} + q_2\sqrt{p_2} + \cdots\right)$ and refer to it as the *BH* degeneracy associated with the area eigenvalue $q_1\sqrt{p_1} + q_2\sqrt{p_2} + \cdots$. Notice that $D^{\mathrm{DL}}(0) = 1$. This will take care of the 1 introduced in the definition of the entropy as $\log(1 + N(a))$. By using Cauchy' s theorem we can write

我们将这个数记为 $D^{\mathrm{DL}}\left(q_1\sqrt{p_1} + q_2\sqrt{p_2} + \cdots\right)$ ，称其为面积本征值 $q_1\sqrt{p_1} + q_2\sqrt{p_2} + \cdots$ 对应的 *BH* 简并度。注意 $D^{\mathrm{DL}}(0) = 1$ 。这会处理熵定义中引入的、作为 $\log(1 + N(a))$ 的常数 1。利用柯西定理，我们可以写出

$$\left[z^0\right] G^{\mathrm{DL}}(z, x_1, \ldots) = \frac{1}{2\pi i} \oint_c \frac{\mathrm{d}z}{z} G^{\mathrm{DL}}(z, x_1, \ldots) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\theta G^{\mathrm{DL}}\left(e^{i\theta}, x_1, \ldots\right),$$

where c is a positively oriented contour around the origin that we parameterize as $z = e^{i\theta}$. For (33) this gives

其中 c 是环绕原点的正向闭合围道，我们将其参数化为 $z = e^{i\theta}$ 。对于式 (33)，这给出

$$\widetilde{G}^{\mathrm{DL}}(x_1, x_2, \ldots) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{1 - 2\sum\limits_{j=1}^{\infty} \sum\limits_{m=1}^{\infty} \cos\left(k_m^j \theta\right) x_j^{y_m^j}} \mathrm{d}\theta. \tag{34}$$

Step 4 To conclude, in order to compute the entropy $S(a)$ we have to take into account the inequality that appears in its definition. To this end, given a value a of the area, we have to repeat the previous procedure for all the area eigenvalues, smaller than or equal to a , and add the resulting BH degeneracies.

步骤 4 最后，为了计算熵 $S(a)$ ，我们必须考虑其定义中出现的不等式。为此，给定一个面积值 a ，我们需要对所有小于等于 a 的面积本征值重复上述过程，然后将得到的黑洞简并度相加。

If we had a concrete formula giving the eigenvalues a_n of the area operator as a function of $n \in \mathbb{N}$, this task would indeed be very simple: we would first build the generating function $f(z) = \sum\limits_{n=0}^{\infty} D^{\mathrm{DL}}(a_n) z^n$ (maybe as a formal power series) and then consider $f(z)/(1 - z)$ (see [76]). However, as far as we know, no such formula is available. The alternative is to use Laplace transforms as shown in the following. To begin with we can write

如果我们有一个具体公式, 能将面积算符的本征值 a_n 表示为 $n \in \mathbb{N}$ 的函数, 那么这项任务会非常简单: 我们可以先构造生成函数 $f(z) = \sum_{n=0}^{\infty} D^{\mathrm{DL}}(a_n) z^n$ (也许是形式幂级数形式), 再考虑 $f(z)/(1-z)$ (参见文献 [76])。但据我们所知，目前还没有这样的公式。替代方法是使用拉普拉斯变换，如下所示。首先我们可以写出

$$\sum_{\{n:a_n \le a\}} D^{\mathrm{DL}}(a_n) = \int_0^a \sum_{n=1}^{\infty} D^{\mathrm{DL}}(a_n)\,\delta(a' - a_n)\,\mathrm{d}a'.$$

Remembering that $\mathcal{L}\left(\delta_{a_0}, s\right) = e^{-a_0 s}, (a_0 > 0), \mathcal{L}\left(\int_0^a f(a')\,\mathrm{d}a', s\right) = \frac{1}{s}\mathcal{L}(f, s)$, we have

记住 $\mathcal{L}\left(\delta_{a_0}, s\right) = e^{-a_0 s}, (a_0 > 0), \mathcal{L}\left(\int_0^a f(a')\,\mathrm{d}a', s\right) = \frac{1}{s}\mathcal{L}(f, s)$，我们得到

$$\sum_{\{n:a_n \le a\}} D^{\mathrm{DL}}(a_n) = \mathcal{L}^{-1}\left(\frac{1}{s}\sum_{n=1}^{\infty} D^{\mathrm{DL}}(a_n)\,e^{-a_n s}, a\right).$$

The key insight now is

现在的核心结论是

$$\begin{aligned}\sum_{n=1}^{\infty} D^{\mathrm{DL}}(a_n)\,e^{-a_n s} &= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{1 - 2\sum_{j=1}^{\infty}\sum_{m=1}^{\infty} e^{-s y_m^j \sqrt{p_j}} \cos\left(k_m^j \theta\right)}\mathrm{d}\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{\left(1 - 2\sum_{k=1}^{\infty} e^{-s\sqrt{k(k+2)}} \cos(k\theta)\right)}\mathrm{d}\theta\end{aligned}$$

which, immediately, gives

它可以直接给出

$$e^{S(a)} = \frac{1}{4\pi^2 i}\int_0^{2\pi}\int_{x_0 - i\infty}^{x_0 + i\infty} \frac{e^{as}}{s\left(1 - 2\sum_{k=1}^{\infty} e^{-s\sqrt{k(k+2)}} \cos(k\theta)\right)}\mathrm{d}s\,\mathrm{d}\theta, \tag{35}$$

where the integration contour, formally denoted with the limits $x_0 - i\infty$ and $x_0 + i\infty$ in the s-integration, is the straight line, parallel to the imaginary axis, $\{z : \mathrm{Re}(z) = x_0\}$ with x_0 larger than the real part of every singularity in the integrand of (35).

其中对 s 积分的积分围道正式用上下限 $x_0 - i\infty$ 和 $x_0 + i\infty$ 表示，是平行于虚轴的直线 $\{z : \mathrm{Re}(z) = x_0\}$，且 x_0 大于式 (35) 被积函数所有奇点的实部。

Several comments are in order now:

现在我们给出几点说明:

- The expression (35) is exact, but it must be remembered that, to compute S for a value of the area spectrum, it is necessary to take a limit from the right.

> - 表达式 (35) 是精确的，但需要注意，要计算面积谱某个值对应的 S ，必须从右侧取极限。

- It is especially useful to study the asymptotic behavior of $S(a)$ in the large- a regime. To this end, it is possible to employ well-known asymptotic techniques. When these are applied to the present case, the Bekenstein-Hawking area law is recovered for a particular choice of the Immirzi parameter $\gamma = 0.237\cdots$.

> - 研究大 a 区域下 $S(a)$ 的渐近行为时，这个表达式尤其有用。对此可以使用成熟的渐近方法。将这些方法应用到当前情况，当伊米尔齐参数取特定值 $\gamma = 0.237\cdots$ 时，可以得到贝肯斯坦-霍金面积定律。

- From a practical point of view, the best way to get concrete values of the entropy for not-too-large areas is to use (34).

> - 从实用角度来看，对于面积不太大的情况，使用式 (34) 得到熵的具体数值是最优方法。

- In order to simplify the computations to understand the behavior of the BH entropy, it is quite common to sidestep the introduction of the projection constraint and work directly with (32). This leads to an approximate value of the entropy given by

> - 为了简化计算、分析黑洞熵的行为，通常会跳过投影约束的引入，直接使用式 (32) 计算。这会得到熵的近似值:

$$e^{S^*(a)} = \frac{1}{2\pi i}\int_{x_0-i\infty}^{x_0+i\infty} \frac{e^{as}}{s\left(1-2\sum\limits_{k=1}^{\infty} e^{-s\sqrt{k(k+2)}}\right)}\mathrm{d}s,$$

which is somewhat easier to analyze than (35). As a matter of fact, this expression leads to the Bekenstein-Hawking law for the same value of $\gamma = 0.237\cdots$, but with no logarithmic corrections.

> 该式比式 (35) 更容易分析。事实上，这个表达式在 $\gamma = 0.237\cdots$ 取相同值时也能得到贝肯斯坦-霍金定律，但不会给出对数修正项。

- A similar approach can be followed to work with some other prescriptions [29, 77]. The main difference from the ones discussed here, as far as the combinatorial problem to be solved is concerned, lies in the form of the projection constraint that must be implemented. In any case, the procedure is similar to the one discussed here: introduce an extra variable z in the relevant generating functions.

> - 类似方法也可用于其他一些规则 [29, 77]。就待解决的组合问题而言，与本文所讨论方法的主要区别在于必须施加的投影约束的形式。无论如何，步骤都与本文所述类似: 在相关生成函数中引入额外变量 z 。

The specific way to do this can be found in [75]. In these two instances the value of the Immirzi parameter leading to the Bekestein-Hawking law is $\gamma = 0.274\cdots$ which differs from the one given above. The respective log corrections are

具体实现方式可见文献 [75]。在这两种情况下，得出贝肯斯坦-霍金定律的伊米尔齐参数值为 $\gamma = 0.274\cdots$，与上文给出的参数值不同。对应的对数修正为

$$-\frac{1}{2}\log\left(a/\ell_P^2\right)\ (\mathrm{GM}),\ -\frac{3}{2}\log\left(a/\ell_P^2\right)\ (\mathrm{ENP}).$$

As mentioned above, the logarithmic term for the $U(1)$ case is also $-\frac{1}{2}\log\left(a/\ell_P^2\right)$.

如上所述，$U(1)$ 情形下的对数项也为 $-\frac{1}{2}\log\left(a/\ell_P^2\right)$。

Features of the Black Hole Degeneracy Spectrum

黑洞简并谱的特征

Very soon after BHs were considered in the context of LQG, direct numerical investigations unearthed an unexpected regularity in the behavior of the entropy as a function of area [21, 22, 78] ; in fact, the entropy appeared to be quantized much in the way predicted by Bekenstein and Mukhanov [79]. One of the first applications of the combinatorial and number theoretic methods described above was to check and confirm the above claims. They were later used to understand the origin of the observed substructure from the first principles.

在圈量子引力 (LQG) 框架下讨论黑洞 (BH) 后不久，直接数值研究就发现，熵作为面积 [21, 22, 78] 的函数表现出了意想不到的规律性；事实上，熵似乎恰好如贝肯斯坦和穆哈诺夫 [79] 所预言那样是量子化的。上文所述组合与数论方法的最早应用之一，就是检验并证实了上述结论。这些方法后来被用于从第一性原理出发，理解观测到的子结构的起源。

The main tool for this purpose is to use the peak counter first proposed in [78]. The idea is to find a way to partition the space of possible configurations of k -labels in such a way that the peaks observed in the BH degeneracy distribution (see Figure 3 of [78]) are isolated. This can be achieved by introducing functions in the space of BH configurations (i.e., the different choices of spin labels for the edges of the spin network state that pierce the horizon) in such a way that their level sets select those corresponding to the peaks in the BH degeneracy spectrum.

用于该研究的核心工具是文献 [78] 最早提出的峰计数器。其思路是找到一种方法划分 k 标记的可能构型空间，从而分离出黑洞简并分布中观测到的峰 (见 [78] 图 3)。这可以通过在黑洞构型空间 (即刺穿视界的自旋网络态各边的不同自旋标记选择) 引入函数实现，使得函数的水平集恰好筛选出对应黑洞简并谱中峰的那些构型。

Given a configuration $\{(k, n_k)\}$, we define the following functions:

给定构型 $\{(k, n_k)\}$，我们定义如下函数:

$$N : \mathcal{C} \to \mathbb{N} : \{(k, n_k)\} \to \sum_k n_k, K : \mathcal{C} \to \mathbb{N} : \{(k, n_k)\} \to \sum_k k n_k, P = 3K + 2N.$$

The first one counts the number of edges of the spin network state that pierce the horizon, the second adds twice the spin labels of these edges, and the third is a simple combination of both of them. The level sets $\mathcal{P}_p := P^{-1}(p)$ of the function P provide a partition of the space of configurations $\mathcal{C} = \cup_{p\in\mathbb{N}}\mathcal{P}_p$. An interesting observation [78] is that for each $p \in \mathbb{N}$ the level set $\mathcal{P}_p$ picks configurations that select a single peak in the BH degeneracy spectrum. This is a nontrivial fact because other conceivable choices - for instance, the functions $P_{(\alpha,\beta)} := \alpha K + \beta N$ with $\alpha, \beta \in \mathbb{N}$ - do not provide such a neat partition of $\mathcal{C}$ (this is discussed in [24]).

第一个函数统计自旋网络态中刺穿视界的边数，第二个函数是两倍的这些边的自旋标记之和，第三个则是前两个的简单组合。函数 P 的水平集 $\mathcal{P}_p := P^{-1}(p)$ 给出了构型空间 $\mathcal{C} = \cup_{p\in\mathbb{N}}\mathcal{P}_p$ 的一个划分。文献 [78] 的一个有趣发现是，对任意 $p \in \mathbb{N}$，水平集 $\mathcal{P}_p$ 筛选出的构型恰好对应黑洞简请谱中的单个峰。这并非平凡结论，因为其他可能的选择——例如函数 $P_{(\alpha,\beta)} := \alpha K + \beta N$ 和 $\alpha, \beta \in \mathbb{N}$ ——都无法给出 $\mathcal{C}$ 这种清晰的划分 (相关讨论见 [24])。

The availability of the peak counter P is very helpful to understand the staircase structure of the BH entropy for microscopic BHs and its eventual persistence for large horizon areas. This is so because the function that gives the entropy as a function of the area can be built as a sum of the contributions of the individual steps singled out by P. In practice this is done by using the generating function

峰计数器 P 的存在非常有助于理解微观黑洞熵的阶梯结构，以及该结构在大视界面积下能否持续存在。这是因为，描述熵随面积变化的函数，可以表示为 P 分离出的各个阶梯的贡献之和。实际计算中我们通过生成函数完成这一点:

$$\widehat{G}^{\mathrm{DL}}(v, s) := \left[z^0\right]\left(1 - \sum_{k=1}^{\infty} v^{3k+2}\left(z^k + z^{-k}\right)e^{-s\sqrt{k(k+2)}}\right)^{-1}, \tag{36}$$

where a new variable v has been introduced in such a way that the inverse Laplace transform of $\left[v^\ell\right]\widehat{G}^{\mathrm{DL}}(v, s)$ gives precisely the contribution of the ℓ -th peak to the BH entropy (see [24] for details). After a suitable normalization, the peaks of the BH degeneracy spectrum can be interpreted as probability densities leading to probability distributions. A remarkable fact is that the values of the parameters that describe these distributions (not only the mean and the variance but also higher moments) can be extracted from the generating function (36) in a straightforward way. Indeed, the expectation value for the n -th power of the area a^n associated with the probability distribution defined by the p -th step in the entropy can be computed as

其中引入了新变量 v，对 $\left[v^\ell\right]\widehat{G}^{\mathrm{DL}}(v, s)$ 做拉普拉斯逆变换就能精确得到第 ℓ 个峰对黑洞熵的贡献 (细节见 [24])。经过适当归一化后，黑洞简并谱的峰可以被解读为概率密度，从而得到概率分布。一个值得注意的结论是，描述这些分布的参数值 (不仅包括均值和方差，还包括更高阶矩) 都可以直接从生成函数 (36) 中提取得到。实际上，熵的第 p 个阶梯定义的概率分布对应的面积 a^n 的 n 次幂期望，可以通过下式计算:

$$E\left(a^n \mid p\right)=(-1)^n \frac{\left[v^p\right]\left(\left.\frac{\partial^n}{\partial s^n}\right|_{s=0} \widehat{G}^{\mathrm{DL}}(v, s)\right)}{\left[v^p\right] \widehat{G}^{\mathrm{DL}}(v, 0)}.$$

From this it is possible to get the mean and the variance of the distribution and find several useful approximations for the BH entropy. The crudest one only makes use of the mean and provides a very good approximation for the position of the observed steps in the entropy when plotted as a function of the horizon area. If the mean and the variance are used, it is possible to approximate the steps as Gaussian distributions. These can be used to understand the fading of the staircase structure as a consequence of the fact that their width grows linearly with the BH area.

由此我们可以得到分布的均值和方差，并得到多个黑洞熵的有用近似。最粗糙的近似仅利用均值，就能很好地拟合出熵作为视界面积函数时，观测到的各个阶梯的位置。如果同时使用均值和方差，我们可以将阶梯近似为高斯分布。利用这些近似可以理解阶梯结构的衰减: 阶梯宽度随黑洞面积线性增长，最终导致结构消失。

We end with a word of caution. In the thermodynamical limit, the entropy satisfies some smoothness and concavity conditions. These are essential in order to use the standard formalism of thermodynamics in which fundamental quantities, such as the temperature or the pressure, are defined as derivatives of the entropy. As discussed in [80], it is important to understand how the thermodynamic limit changes the results discussed here, which have been obtained in the microcanonical ensemble.

最后我们要提请注意。在热力学极限下，熵满足若干光滑性与凹性条件。这些条件是使用标准热力学形式体系的核心——温度、压强等基本量正是在该体系中被定义为熵的导数。正如文献 [80] 所讨论的，我们此处讨论的结果是在微正则系综中得到的，因此理解热力学极限会如何改变这些结果十分重要。

Entropy DOF and the Immirzi Parameter

熵自由度与伊米尔齐参数

Now that we have presented the technical details for the derivation of the Bekenstein-Hawking area law from the counting of microscopic states of the horizon as identified within the LQG framework, let us comment on the nature of these DOF and the related issue of the fixation of the Immirzi parameter.

既然我们已经介绍了在 LQG 框架中通过对视界微观态计数推导贝肯斯坦-霍金面积定律的技术细节，接下来我们讨论这些自由度的本质，以及确定伊米尔齐参数的相关问题。

We reviewed in section ”Constraints and Charges” that the IH phase space - before imposition of spherical symmetry - is characterized by the symmetry group (11). As elucidated in [45,46], different formulations of gravity, related by different choices of boundary Lagrangian, can provide different representations of the corner symmetry group. We saw that in the Palatini formulation, which is the standard starting point of the LQG quantization of IHs, the symmetry group (11) is trivially represented, namely all the IH corner charges vanish. It is only at the quantum level that a finite set of these local SU(2) charges is activated through singular excitations of quantum geometry. In addition, also a subset of tangent diffeomorphisms permuting the punctures is included in the counting. Therefore, from the standard perspective, all the DOF accounting for the BH entropy have a purely quantum origin, while the classical counting for a spherically symmetric IH would naively lead to a zero entropy.

> 我们已在“约束与荷”一节回顾过: 在引入球对称性之前，孤立视界 (IH) 相空间由对称性群 (11) 描述。正如文献 [45,46] 所阐明，通过选择不同边界拉格朗日量得到的不同引力表述，可以给出顶角对称性群的不同表示。我们已经看到，作为 LQG 量子化孤立视界的标准出发点，Palatini 表述中对称性群 (11) 是平凡表示的，即所有 IH 顶角荷都为零。只有在量子层面，量子几何的奇异激发才会激活有限个这类局域 SU(2) 荷。此外，对 punctures 做置换的切向微分同胚的子集也被纳入计数。因此，从标准视角来看，所有贡献黑洞熵的自由度都纯粹是量子起源的，球对称 IH 的经典计数按朴素计算会得到零熵。

This perspective, however, may seem quite counterintuitive from a statistical mechanics point of view applied to ordinary systems, like an ideal gas, where any quantum degree of freedom has a classical counterpart. In fact, the situation is usually the opposite than the one described above, with the Gibbs entropy (the classical analog of the von Neumann entropy like (22)) that is often divergent, since the properties of classical systems are continuous and the number of classical microstates uncountably infinite; it is only a coarse graining of the phase space that renders the classical statistical entropy of the system finite. In the quantum theory, this regularization procedure is implemented through the discreteness of the spectrum of the relevant observable, like the energy, defining the ensemble.

> 然而，从应用于理想气体这类普通系统的统计力学角度来看，这个观点可能相当违背直觉——在普通系统中，任何量子自由度都存在经典对应。实际上，通常情况和上文描述的恰好相反: 吉布斯熵 (即冯·诺依曼熵的经典类似物，如式 (22)) 往往是发散的，因为经典系统的性质是连续的，经典微观态的数量是不可数无穷大；只有对相空间进行粗粒化才能得到系统有限的经典统计熵。在量子理论中，这个正则化过程是通过相关可观测量 (如能量) 谱的离散性来实现的，由此定义系综。

We thus see that the interpretation of the BH entropy DOF can be reconciled with this familiar statistical mechanics point of view if we adopt a gravity formulation, like the ECH one (6), where the IH symmetry group is represented nontrivially in the phase space by having an infinite set of non-vanishing charges at the classical and continuum level. In this case then, the crucial question is whether the quantization of such phase space, or equivalently of the IH corner symmetry algebra, can be achieved and the counting based on the new set of quantum numbers still yields an entropy proportional to the area. While entering this terrain is beyond the scope of this chapter, let us conclude this section with a few observations and remarks about this alternative description of an IH quantum geometry.

> 因此我们可以看到，如果我们采用一种引力表述，例如 ECH 表述 (6)——其中孤立视界的对称群在相空间中是非平凡表示的，在经典和连续水平上存在无穷多非零荷——那么黑洞熵自由度的诠释就可以与我们熟悉的统计力学观点相容。在这种情况下，关键问题在于: 这类相空间的量子化 (等价于孤立视界边界对称性代数的量子化) 能否实现，以及基于新量子数的计数是否仍能得到正比于面积的熵。虽然进入这一领域超出了本章的范围，让我们在本节最后对这种孤立视界量子几何的替代描述做一些观察与评述。

It was shown in [45, 81] that a regularization procedure of the infinite-dimensional corner symmetry algebra, independent of a choice of bulk discretiza-tion, can be introduced, yielding a finite-dimensional coarse-grained subalgebra associated with (11). This allows one to recover a discrete surface area spectrum already at the continuum and semi-classical level, as well as the LQG flux algebra representing one of the main ingredients of the IH quantization we reviewed. Moreover, applying a similar regularization in the time-gauge context, a notion of infinitesimal diffeomorphism operators corresponding to spatial translations on the corner was derived in [44]. By an appropriate choice of smearing vector fields on H (reflecting the IH boundary conditions), the latter could be understood as the generators of puncture reordering at the quantum level. Following this strategy, one could then arrive at a picture where the standard LQG counting presented in section ”Black Holes and Combinatorics” applies as well to a new construction of the IH Hilbert space based on the quantization of a finite-dimensional subalgebra of the IH corner symmetry algebra. This would put the entropy counting in line with the usual treatment of statistical mechanical systems: The infinite number of horizon classical DOF gets regularized by a discrete representation for the choice of the area element on the IH cross-section, which forms a Casimir of its symmetry algebra and thus acts diagonally on irreducible representations of G_H .

> 文献 [45, 81] 表明，可以引入一种独立于体离散化选择的无穷维角对称性代数正则化方案，得到与式 (11) 关联的有限维粗粒化子代数。这使得我们可以在连续统和半经典层面就得到离散的表面积谱，以及我们此前回顾的 IH 量子化核心要素之一——LQG 通量代数。此外，文献 [44] 在时间规范语境下应用了类似的正则化，推导得到了对应角上空间平移的无穷小微分同胚算符概念。通过适当选择 H 上的涂抹矢量场 (反映 IH 边界条件)，后者可以被理解为量子层面刺点重排序的生成元。遵循这一思路，我们可以得到这样一个图景:“黑洞与组合学”一节中介绍的标准 LQG 计数，同样适用于基于 IH 角对称性代数有限维子代数量子化构造的全新 IH 希尔伯特空间。这会让熵计数符合统计力学系统的常规处理方式: 视界经典自由度的无穷多个自由度，通过选择 IH 截面上的面积元得到离散表示正则化，面积元是对称性代数的卡西米尔量，因此在 G_H 的不可约表示上对角作用。

The other advantage of this alternative treatment of the IH phase space is the fact that the role of γ becomes apparent already at the semi-classical level. In fact, while leaving the classical bulk dynamics unaffected, only in the presence of a non-vanishing γ we have access to a nontrivial representation of the internal Lorentz transformations on the corner phase space. Moreover, γ appears as a proportionality constant between the surface area element and the SU(2) Casimir already at the semi-classical and continuum levels [45, 81]; this provides a pre-quantization evidence of how its numerical value labels unitarily inequivalent Irreps of the corner symmetry group defining the non-radiative (kinematical) Hilbert space of the theory. From this perspective then, it is clear how γ plays a crucial role already at the semi-classical level and the fact that its numerical value needs to be fixed in order to recover the entropy-area law becomes a natural feature of the approach that we have followed.

> 对孤立视界相空间的这种替代处理的另一优势在于: γ 的作用在半经典层面就已经显现。事实上，它不会改变经典体动力学，且只有当 γ 非零时，我们才能得到边界相空间上内部洛伦兹变换的非平庸表示。此外，即便在半经典和连续层面，γ 也表现为表面积元与 SU(2) 卡西米尔量之间的比例常数 [45, 81]；这为其数值如何标记定义了理论非辐射 (运动学) 希尔伯特空间的边界对称群的酉不等价不可约表示提供了量子化前的证据。由此视角可以清晰看出，γ 在半经典层面就已经起到关键作用，因此需要固定其数值才能得到贝肯斯坦-霍金熵面积定律，这是我们所采用方法的一个自然性质。

While these considerations can resolve the possible tension caused by the numerical fixation of γ , they surely do not disqualify previous attempts to eliminate it or at least alleviate it. Some of these include:

这些考量可以解决固定 γ 数值可能带来的矛盾，但它们确实没有否定此前消除或至少弱化该问题的尝试，这些尝试包括:

- Considering the running from the UV to the IR of Newton's constant and the horizon area within the effective expression for the entropy [82].

- 在熵的有效表达式中考虑牛顿常数与视界面积从紫外到红外的跑动 [82]。

- Introduction of a new parameterization of the boundary connection independent from the Immirzi parameter in the bulk [31, 33] .

- 引入不依赖体 [31, 33] 即伊米尔齐参数的边界联络新参数化方案。

- Modification of the first law of BH mechanics through the introduction of a new quantum hair associated with the number of punctures [83].

- 通过引入与穿孔数相关的新量子毛修正黑洞力学第一定律 [83]。

- Construction of a local quantum Rindler horizon generated by the boost Hamiltonian of Lorentzian Spinfoams [84].

- 构造由洛伦兹自旋泡沫 boost 哈密顿量生成的局域量子林德勒视界 [84]。

- Analytic continuation to $\gamma = i$ in order to restore the full spacetime covariance of the Ashtekar connection and a proper notion of horizon thermality [85-89].

- 对 $\gamma = i$ 做解析延拓，以恢复阿西卡联络的完整时空协变性以及视界热平衡的恰当定义 [85-89]。

- Introduction of an extra holographic degeneracy factor in the IH partition function associated with the entanglement of matter DOF near the horizon [65].

- 在孤立视界配分函数中引入额外的全息简并因子，对应视界附近物质自由度的纠缠 [65]。

- Construction of condensate states in the Group Field Theory formalism, encoding the continuum spherically symmetric quantum geometry of a horizon [69, 90].

- 在群场论形式体系中构造凝聚态，编码视界的连续球对称量子几何 [69, 90]。

In the end, the correctedness of the value of the Immirzi parameter predicted by the standard LQG BH entropy calculation can be addressed in a conclusive manner only through observational tests sensitive to the area gap; for promising steps in this direction within a cosmological setting, see [91] and the Chap. 90, ”Loop Quantum Cosmology: Relation Between Theory and Observations.” Alternatively or (hopefully) in addition to this path, one can hope to have at least another independent theoretical model descending as close as possible from the full LQG framework, where the same numerical value is predicted by demanding a given outcome or value for an observable of physical relevance. In this regard, it is intriguing to point out that the value of $\gamma = 0.274\cdots$ obtained from the SU(2) counting has been predicted from the study of the effective dynamics describing a Schwarzschild BH interior as derived from a partial gauge fixing of the full loop quantum gravity Hilbert space; in this model [92], the physical relevance of the specific numerical value is related to the behavior of the post-bounce interior geometry, which approaches an asymptotically de Sitter geometry only for that specific value - see the Chap. 92, ”Quantum Geometry and Black Holes” for other constructions of BH interior effective geometries in LQG.

最终，标准圈量子引力黑洞熵计算预测的伊米尔齐参数值是否正确，只能通过对面积间隙敏感的观测检验给出定论; 关于宇宙学背景下该方向的前沿进展, 参见 [91] 和第 90 章“圈量子宇宙学: 理论与观测的关系”。除此以外，(我们也希望) 还可以得到另一个独立理论模型，它尽可能从完整圈量子引力框架出发，通过要求得到某一物理相关可观测量的给定结果或数值来预测出相同的 $\gamma = 0.274\cdots$ 值。在此方面值得注意的是，通过 SU(2) 计数得到的 $\gamma = 0.274\cdots$ 值，已经可以从描述史瓦西黑洞内部的有效动力学研究中预测得到，该动力学是对完整圈量子引力希尔伯特空间做部分规范固定得到的；在这个模型 [92] 中，该特定数值的物理相关性与反弹后内部几何的行为有关，只有取这个值时反弹后的几何才会渐近趋近德西特几何——关于圈量子引力中黑洞内部有效几何的其他构造，参见第 92 章“量子几何与黑洞”。

Cross-References

交叉引用

Emergence of Riemannian Quantum Geometry

黎曼量子几何的涌现

- Loop Quantum Cosmology: Relation Between Theory and Observations

  - 圈量子宇宙学: 理论与观测的关联

- Quantum Geometry and Black Holes

  - 量子几何与黑洞

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